Contents

1 Introduction

These brief notes include major definitions and theorems of the graph theory lecture held by Prof. Maria Axenovich at KIT in the winter term 2013/14. We neither prove nor motivate the results and definitions. You can look up the proofs of the theorems in the book "Graph Theory" by Reinhard Diestel [4]. A free version of the book is available at `http://diestel-graph-theory.com`.

Conventions:

- $G = (V, E)$ is an arbitrary (undirected, simple) graph
- $n := |V|$ is its number of vertices
- $m := |E|$ is its number of edges

2 Notations

notation	definition	meaning
$\binom{V}{k}$, V finite set, k integer	$\{S \subseteq V : \|S\| = k\}$	the set of all k-element subsets of V
V^2, V finite set	$\{(u, v) : u, v \in V\}$	the set of all ordered pairs of elements in V
$[n]$, n integer	$\{1, \ldots, n\}$	the set of the first n positive integers
$\mathbb{N}$	$1, 2, \ldots$	the natural numbers, not including 0
2^S, S finite set	$\{T : T \subseteq S\}$	the power set of S, i.e., the set of all subsets of S
$S \triangle T$, S, T finite sets	$(S \cup T) \setminus (S \cap T)$	the symmetric difference of sets S and T, i.e., the set of elements that appear in exactly one of S or T
$A \dot{\cup} B$, A, B disjoint sets	$A \cup B$	the disjoint union of A and B

3 Preliminaries

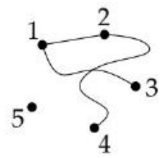
Definition. A *graph* G is an ordered pair (V, E), where V is a finite set and $E \subseteq \binom{V}{2}$ is a set of pairs of elements in V. graph, G

- The set V is called the set of *vertices* and E is called the set of *edges* of G. vertex, edge
- The edge $e = \{u, v\} \in \binom{V}{2}$ is also denoted by $e = uv$.
- If $e = uv \in E$ is an edge of G, then u is called *adjacent* to v and u is called *incident* to e. adjacent, incident
- If e_1 and e_2 are two edges of G, then e_1 and e_2 are called *adjacent* if $e_1 \cap e_2 \neq \emptyset$, i.e., the two edges are incident to the same vertex in G.

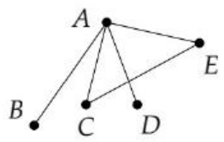
We can visualize graphs $G = (V, E)$ using pictures. For each vertex $v \in V$ we draw a point (or small disc) in the plane. And for each edge $uv \in E$ we draw a continuous curve starting and ending in the point/disc for u and v, respectively.

Several examples of graphs and their corresponding pictures follow:

$$V = [5],\ E = \{12, 13, 24\}$$



$$V = \{A, B, C, D, E\},$$
$$E = \{AB, AC, AD, AE, CE\}$$



Definition (Graph variants).

- A *directed graph* is a pair $G = (V, A)$ where V is a finite set and $E \subseteq V^2$. The edges of a directed graph are also called *arcs*. directed graph, arc
- A *multigraph* is a pair $G = (V, E)$ where V is a finite set and E is a multiset of elements from $\binom{V}{1} \cup \binom{V}{2}$, i.e., we also allow loops and multiedges. multigraph
- A *hypergraph* is a pair $H = (X, E)$ where X is a finite set and $E \subseteq 2^X \setminus \{\emptyset\}$. hypergraph

Definition. For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ we say that G_1 and G_2 are *isomorphic*, denoted by $G_1 \simeq G_2$, if there exists a bijection $\phi : V_1 \to V_2$ with $xy \in E_1$ if and only if $\phi(x)\phi(y) \in E_2$. Loosely speaking, G_1 and G_2 are isomorphic if they are the same up to renaming of vertices. isomorphic, $\simeq$

When making structural comments, we do not normally distinguish between isomorphic graphs. Hence, we usually write $G_1 = G_2$ instead of $G_1 \simeq G_2$ whenever vertices $=$

are indistinguishable. Then we use the informal expression *unlabeled graph* (or just *graph* when it is clear from the context) to mean an isomorphism class of graphs.

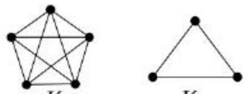
unlabeled graph

Important graphs and graph classes

Definition. For all natural numbers n we define:

- the *complete graph* K_n on n vertices as the (unlabeled) graph isomorphic to $\big([n], \binom{[n]}{2}\big)$. Complete graphs correspond to *cliques*.

  complete graph, K_n

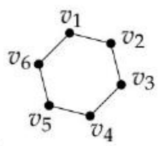


K_5 K_3

- for $n \geq 3$, the *cycle* C_n on n vertices as the (unlabeled) graph isomorphic to $\big([n], \big\{\{i, i+1\} : i = 1, \ldots, n-1\big\} \cup \big\{n, 1\big\}\big)$. The *length of a cycle* is its number of edges. We write $C_n = 12 \ldots n1$. The cycle of length 3 is also called a *triangle*.

  cycle, C_n

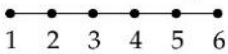
  triangle



$C_6 = v_1v_2v_3v_4v_5v_6v_1$

- the *path* P_n on n vertices as the (unlabeled) graph isomorphic to $\big([n], \big\{\{i, i+1\} : i = 1, \ldots, n-1\big\}\big)$. The vertices 1 and n are called the *endpoints* or *ends* of the path. The *length of a path* is its number of edges. We write $P_n = 12 \ldots n$.

  path, P_n



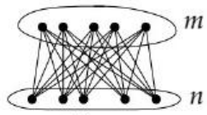
- the *empty graph* E_n on n vertices as the (unlabeled) graph isomorphic to $\big([n], \emptyset\big)$. Empty graphs correspond to independent sets.

  empty graph, E_n

E_{10}

- for $m \geq 1$, the *complete bipartite graph* $K_{m,n}$ on $n+m$ vertices as the (unlabeled) graph isomorphic to $(A \cup B, \{xy : x \in A, y \in B\})$, where $|A| = m$ and $|B| = n$, $A \cap B = \emptyset$.

  complete bipartite graph, $K_{m,n}$



$K_{m,n}$

- for $r \geq 2$, a *complete r-partite* graph as an (unlabeled) graph isomorphic to

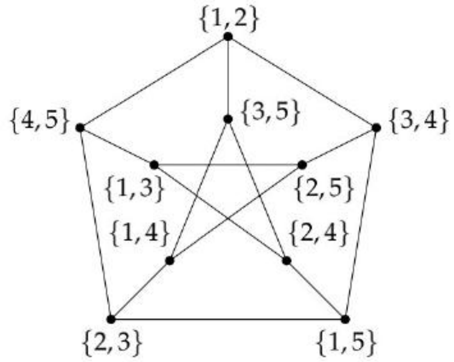
complete r-partite

$$\big(A_1 \dot\cup \cdots \dot\cup A_r, \{xy : x \in A_i, y \in A_j, i \neq j\}\big),$$

where $A_1, \ldots, A_r$ are non-empty finite sets. In particular, the complete bipartite graph $K_{m,n}$ is a complete 2-partite graph.

- the *Petersen graph* as the (unlabeled) graph isomorphic to $\big(\binom{[5]}{2}, \big\{\{S,T\} : S,T \in \binom{[5]}{2}, S \cap T = \emptyset\big\}\big)$.

Petersen graph



- for a natural number k, $k \leq n$, the *Kneser graph* $K(n,k)$ as the (unlabeled) graph isomorphic to

Kneser graph, $K(n,k)$

$$\left(\binom{[n]}{k}, \left\{\{S,T\} : S,T \in \binom{[n]}{k}, S \cap T = \emptyset\right\}\right).$$

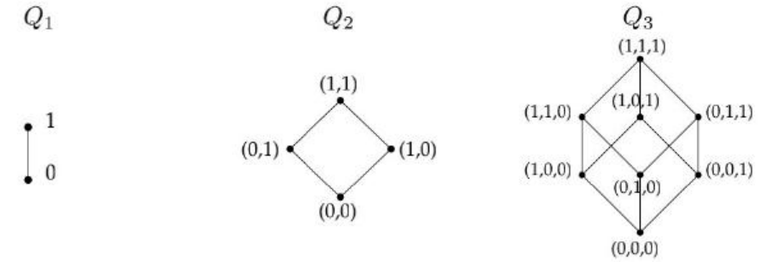
Note that $K(5,2)$ is the Petersen graph.

- the *n-dimensional hypercube* Q_n as the (unlabeled) graph isomorphic to

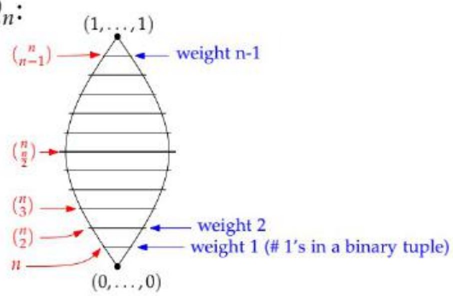
hypercube, Q_n

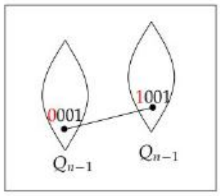
$$\big(2^{[n]}, \big\{\{S,T\} : S,T \in 2^{[n]}, |S \triangle T| = 1\big\}\big).$$

Vertices are labeled either by corresponding sets or binary indicators vectors. For example the vertex $\{1,3,4\}$ in Q_6 is coded by $(1,0,1,1,0,0,0)$.



Q_n:



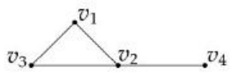


Basic graph parameters and degrees

Definition. Let $G = (V, E)$ be a graph. We define the following parameters of G.

- The graph G is *non-trivial* if it contains at least one edge, i.e., $E \neq \emptyset$. Equivalently, G is non-trivial if G is not an empty graph. — non-trivial
- The *order of* G, denoted by $|G|$, is the number of vertices of G, i.e., $|G| = |V|$. — order, $|G|$
- The *size of* G, denoted by $\|G\|$, is the number of edges of G, i.e., $\|G\| = |E|$. Note that if the order of G is n, then the size of G is between 0 and $\binom{n}{2}$. — size, $\|G\|$
- Let $S \subseteq V$. The *neighbourhood of* S, denoted by $N(S)$, is the set of vertices in V that have an adjacent vertex in S. The elements of $N(S)$ are called *neighbours* of S. Instead of $N(\{v\})$ for $v \in V$ we usually write $N(v)$. — neighbourhood, $N(v)$; neighbour
- If the vertices of G are labeled $v_1, \ldots, v_n$, then there is an $n \times n$ matrix A with entries in $\{0, 1\}$, which is called the *adjacency matrix* and is defined as follows: — adjacency matrix

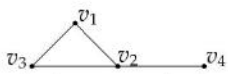
$$v_i v_j \in E \qquad \Leftrightarrow \qquad A[i, j] = 1$$



$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

A graph and its adjacency matrix.

- The *degree* of a vertex v of G, denoted by $d(v)$ or $\deg(v)$, is the number of edges incident to v. — degree, $d(v)$



$\deg(v_1) = 2$, $\deg(v_2) = 3$, $\deg(v_3) = 2$, $\deg(v_4) = 1$

- A vertex of degree 1 in G is called a *leaf*, and a vertex of degree 0 in G is called an *isolated vertex*. — leaf; isolated vertex
- The *degree sequence* of G is the multiset of degrees of vertices of G, e.g. in the example above the degree sequence is $\{1, 2, 2, 3\}$. — degree sequence
- The *minimum degree of* G, denoted by $\delta(G)$, is the smallest vertex degree in G (it is 1 in the example). — minimum degree, $\delta(G)$
- The *maximum degree of* G, denoted by $\Delta(G)$, is the highest vertex degree in G (it is 3 in the example). — maximum degree, $\Delta(G)$
- The graph G is called *k-regular* for a natural number k if all vertices have degree k. Graphs that are 3-regular are also called *cubic*. — regular; cubic
- The *average degree of* G is defined as $d(G) = \left(\sum_{v \in V} \deg(v)\right)/|V|$. Clearly, we have $\delta(G) \leq d(G) \leq \Delta(G)$ with equality if and only if G is k-regular for some k. — average degree, $d(G)$

Lemma 1 (Handshake Lemma, 1.2.1)**.** For every graph $G = (V, E)$ we have

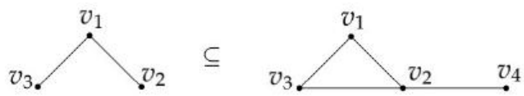
$$2|E| = \sum_{v \in V} d(v).$$

Corollary 2. The sum of all vertex degrees is even and therefore the number of vertices with odd degree is even.

Subgraphs

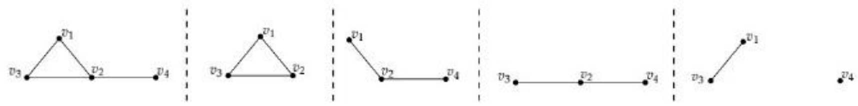
Definition.

- A graph $H = (V', E')$ is a *subgraph of* G, denoted by $H \subseteq G$, if $V' \subseteq V$ and $E' \subseteq E$. If H is a subgraph of G, then G is called a *supergraph of* H, denoted by $G \supseteq H$. In particular, $G_1 = G_2$ if and only if $G_1 \subseteq G_2$ and $G_1 \supseteq G_2$. — subgraph, $\subseteq$; supergraph, $\supseteq$



- A subgraph H of G is called an *induced subgraph* of G if for every two vertices $u, v \in V(H)$ we have $uv \in E(H) \Leftrightarrow uv \in E(G)$. In the example above H is not an induced subgraph of G. Every induced subgraph of G can be obtained by deleting vertices (and all incident edges) from G. — induced subgraph

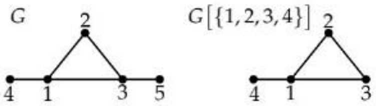
  Examples:



- Every induced subgraph of G is uniquely defined by its vertex set. We write

$G[X]$ for the induced subgraph of G on vertex set X, i.e., $G[X] = \big(X, \{xy : x, y \in X,\ xy \in E(G)\}\big)$. Then $G[X]$ is called the *subgraph of G induced by the vertex set $X \subseteq V(G)$*. $G[X]$

Example:



- If H and G are two graphs, then an *(induced) copy* of H in G is an (induced) subgraph of G that is isomorphic to H. copy
- A subgraph $H = (V', E')$ of $G = (V, E)$ is called a *spanning subgraph* of G if $V' = V$. spanning subgraph
- A graph $G = (V, E)$ is called *bipartite* if there exists natural numbers m, n such that G is isomorphic to a subgraph of $K_{m,n}$. In this case, the vertex set can be written as $V = A \dot{\cup} B$ such that $E \subseteq \{ab \mid a \in A, b \in B\}$. The sets A and B are called *partite sets of G*. bipartite, partite sets
- A *cycle (path, clique)* in G is a subgraph H of G that is a cycle (path, complete graph). clique
- An *independent set* in G is an induced subgraph H of G that is an empty graph. independent set
- A *walk* (of length k) is a non-empty alternating sequence $v_0 e_0 v_1 e_1 \cdots e_{k-1} v_k$ of vertices and edges in G such that $e_i = \{v_i, v_{i+1}\}$ for all $i < k$. If $v_0 = v_k$, the walk is *closed*. walk, closed walk
- Let $A, B \subseteq V$, $A \cap B = \emptyset$. A path P in G is called an *A-B-path* if $P = v_1 \dots v_k$, $V(P) \cap A = \{v_1\}$ and $V(P) \cap B = \{v_k\}$. When $A = \{a\}$ and $B = \{b\}$, we simply call P an a-b-path. If G contains an a-b-path we say that the vertices a and b are *linked by a path*. A-B-path
- Two paths P, P' in G are called *independent* if every vertex contained in both P and P' (if any) is an endpoint of P and P'. I.e., P and P' can share only endpoints. independent paths
- A graph G is called *connected* if any two vertices are linked by a path. connected
- A subgraph H of G is *maximal*, respectively *minimal*, with respect to some property if there is no supergraph, respectively subgraph, of H with that property. maximal, minimal
- A maximal connected subgraph of G is called a *connected component* of G. component
- A graph G is called *acyclic* if G does not have any cycle. Acyclic graphs are also called *forests*. acyclic, forest
- A graph G is called a *tree* if G is connected and acyclic. tree

Proposition 3. If a graph G has minimum degree $\delta(G) \geq 2$, then G has a path of length $\delta(G)$ and a cycle with at least $\delta(G)+1$ vertices.

Proposition 4. If a graph has a u-v-walk, then it has a u-v-path.

Proposition 5. If a graph has a closed walk of odd length, then it contains an odd cycle.

Proposition 6. If a graph has a closed walk with a non-repeated edge, then the graph contains a cycle.

Proposition 7. A graph is bipartite if and only if it has no cycles of odd length.

Definition. An *Eulerian tour of G* is a closed walk containing all edges of G, each exactly once.

Eulerian tour

Theorem 8 (Eulerian Tour Condition, 1.8.1)**.** A connected graph has an Eulerian tour if and only if every vertex has even degree.

Lemma 9. Every tree on at least two vertices has a leaf.

Lemma 10. A tree of order $n \geq 1$ has exactly $n-1$ edges.

Lemma 11. Every connected graph contains a spanning tree.

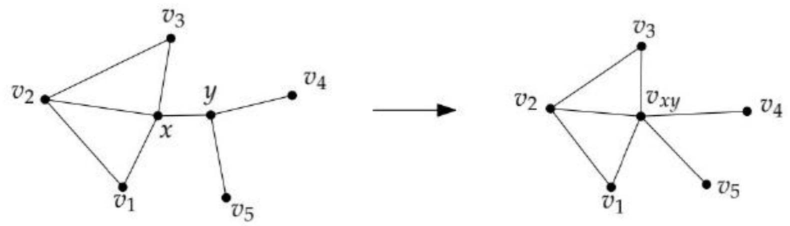
Lemma 12. A connected graph on $n \geq 1$ vertices and $n-1$ edges is a tree.

Lemma 13. The vertices of every connected graph can be ordered $(v_1, \ldots, v_n)$ so that for every $i \in \{1, \ldots, n\}$ the graph $G\big[\{v_1, \ldots, v_i\}\big]$ is connected.

Operations on graphs

Definition. Let $G = (V, E)$ and $G' = (V', E')$ be two graphs, $U \subseteq V$ be a subset of vertices of G and $F \subseteq \binom{V}{2}$ be a subset of pairs of vertices of G. Then we define

- $G \cup G' := (V \cup V', E \cup E')$ and $G \cap G' := (V \cap V', E \cap E')$. Note that $G, G' \subseteq G \cup G'$ and $G \cap G' \subseteq G, G'$. Sometimes, we also write $G + G'$ for $G \cup G'$. — $G \cup G', G \cap G'$
- $G - U := G[V \setminus U]$, $G - F := (V, E \setminus F)$ and $G + F := (V, E \cup F)$. If $U = \{u\}$ or $F = \{e\}$ then we simply write $G - u$, $G - e$ and $G + e$ for $G - U$, $G - F$ and $G + F$, respectively. — $G - U$, $G - F$, $G + F$
- For an edge $e = xy$ in G we define $G \circ e$ as the graph obtained from G by identifying x and y and removing (if necessary) loops and multiple edges. We say that $G \circ e$ arises from G by *contracting the edge e*. — $G \circ e$; contract



- The *complement of* G, denoted by $\overline{G}$ or G^C, is defined as the graph $(V, \binom{V}{2} \setminus E)$. In particular, $G + \overline{G}$ is a complete graph, and $\overline{G} = (G + \overline{G}) - E$. complement, $\overline{G}$

More graph parameters

Definition. Let $G = (V, E)$ be any graph.

- The *girth of* G, denoted by $g(G)$, is the length of a shortest cycle in G. If G is acyclic, its girth is said to be ∞. girth, $g(G)$
- The *circumference* of G is the length of a longest cycle in G. If G is acyclic, its circumference is said to be 0. circumference
- The graph G is called *Hamiltonian* if G has a spanning cycle, i.e., there is a cycle in G that contains every vertex of G. In other words, G is Hamiltonian if and only if its circumference is $|V|$. Hamiltonian
- The graph G is called *traceable* if G has a spanning path, i.e., there is a path in G that contains every vertex of G. traceable
- For two vertices u and v in G, the *distance between u and v*, denoted by $d(u, v)$, is the length of a shortest u-v-path in G. If no such path exists, $d(u, v)$ is said to be ∞. distance, $d(u, v)$
- The *diameter of* G, denoted by $\text{diam}(G)$, is the maximum distance among all pairs of vertices in G, i.e. diameter, $\text{diam}(G)$

$$\text{diam}(G) = \max_{u,v \in V} d(u, v).$$

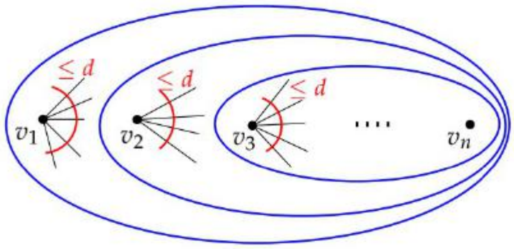
- The *radius of* G, denoted by $\text{rad}(G)$, is defined as radius, $\text{rad}(G)$

$$\text{rad}(G) = \min_{u \in V} \max_{v \in V} d(u, v).$$

- If there is a vertex ordering $v_1, \ldots, v_n$ of G and a $d \in \mathbb{N}$ such that

$$|N(v_i) \cap \{v_{i+1}, \ldots, v_n\}| \le d,$$

for all $i \in [n-1]$ then G is called *d-degenerate*. The minimum d for which G is d-degenerate is called the *degeneracy* of G. d-degenerate degeneracy



We remark that the 1-degenerate graphs are precisely the forests.

- A *proper k-edge colouring* is an assignment $c'\colon E \to [k]$ of colours in $[k]$ to edges such that no two adjacent edges receive the same colour. The *chromatic index of G*, or *edge-chromatic number*, is the minimal k such that G has a k-edge colouring. It is denoted by $\chi'(G)$. proper edge colouring; chromatic index, $\chi'(G)$
- A *proper k-vertex colouring* is an assignment $c\colon V \to [k]$ of colours in $[k]$ to vertices such that no two adjacent vertices receive the same colour. The *chromatic number of G* is the minimal k such that G has a k-vertex colouring. It is denoted by $\chi(G)$. proper vertex colouring; chromatic number, $\chi(G)$

Proposition 14. For any graph $G = (V, E)$ the following are equivalent:

(i) G is a tree, that is, G is connected and acyclic.

(ii) G is connected, but for any edge $e \in E$ in G the graph $G - e$ is not connected.

(iii) G is acyclic, but for any edge $e \notin E$ not in G the graph $G + e$ has a cycle.

(iv) G is connected and 1-degenerate.

(v) G is connected and $|E| = |V| - 1$.

(vi) G is acyclic and $|E| = |V| - 1$.

(vii) G is connected and every non-trivial subgraph of G has a vertex of degree at most 1.

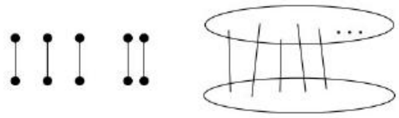
(viii) Any two vertices are joined by a unique path in G.

4 Matchings

Definition.

- A *matching* (*independent edge set*) is a vertex-disjoint union of edges, i.e., the union of pairwise non-adjacent edges.

matching



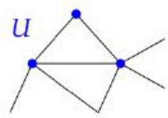
- A *matching in* G is a subgraph of G isomorphic to a matching. We denote the size of the largest matching in G by $\nu(G)$.

$\nu(G)$

- A *vertex cover in* G is a set of vertices $U \subseteq V$ such that each edge in E is incident to at least one vertex in U. We denote the size of the smallest vertex cover in G by $\tau(G)$.

vertex cover

$\tau(G)$



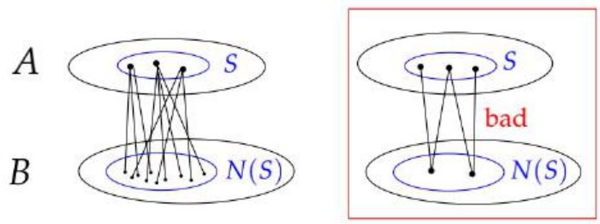
- A k*-factor of* G is a k-regular spanning subgraph of G.

k-factor

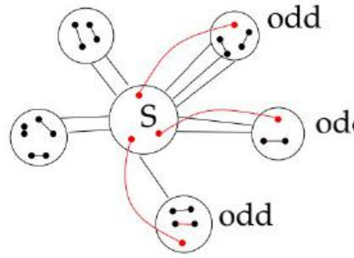
- A *1-factor of* G is also called a *perfect matching* since it is a matching of largest possible size in a graph of order $|V|$. Clearly, G can only contain a perfect matching if $|V|$ is even.

perfect matching

Theorem 15 (Hall's Marriage Theorem, 2.1.2)**.** Let G be a bipartite graph with partite sets A and B. Then G has a matching containing all vertices of A if and only if $|N(S)| \geq |S|$ for all $S \subseteq A$.



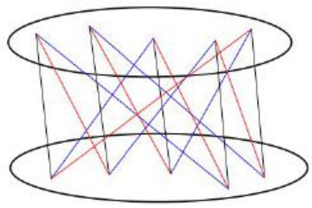
Theorem 16 (Tutte's Theorem, 2.2.1)**.** For $S \subseteq V$ define $q(S)$ to be the number of odd components of $G - S$, i.e., the number of connected components of $G - S$ consisting of an odd number of vertices. A graph G has a perfect matching if and only if $q(S) \leq |S|$ for all $S \subseteq V$.



$|S| \geq$ odd components of $G - S$

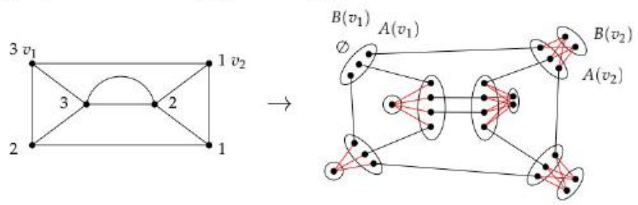
Corollary 17.

- Let G be bipartite with partite sets A and B such that $|N(S)| \geq |S| - d$ for all $S \subseteq A$, and a fixed positive integer d. Then G contains a matching of size at least $|A| - d$.
- A k-regular bipartite graph has a perfect matching.
- A k-regular bipartite graph has a proper k-edge coloring.

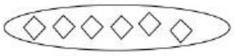


Definition. Let $G = (V, E)$ be any graph.

- For all functions $f\colon V \to \mathbb{N} \cup \{0\}$ an *f-factor of G* is a spanning subgraph H of G such that $\deg_H(v) = f(v)$ for all $v \in V$. f-factor
- Let $f\colon V \to \mathbb{N} \cup \{0\}$ be a function with $f(v) \leq \deg(v)$ for all $v \in V$. We can construct the auxiliary graph $T(G, f)$ by replacing each vertex v with vertex sets $A(v) \cup B(v)$ such that $|A(v)| = \deg(v)$ and $|B(v)| = \deg(v) - f(v)$. For adjacent vertices u and v we place an edge between $A(u)$ and $A(v)$ such that the edges between the A-sets are independent. We also insert a complete bipartite graph between $A(v)$ and $B(v)$ for each vertex v. $T(G, f)$



- Let H be a graph. An *H-factor of G* is a spanning subgraph of G that is a vertex-disjoint union of copies of H, i.e., a set of copies of H in G whose vertex sets form a partition of V. H-factor

$H =$  $G =$ 

Lemma 18. Let $f\colon V \to \mathbb{N} \cup \{0\}$ be a function with $f(v) \leq \deg(v)$ for all $v \in V$. Then G has an f-factor if and only if $T(G, f)$ has a 1-factor.

Theorem 19 (König's Theorem, 2.1.1)**.** Let G be bipartite. Then $\nu(G) = \tau(G)$, i.e., the size of a largest matching is the same as the size of a smallest vertex cover.

Theorem 20 (Hajnal and Szemerédi)**.** If G satisfies $\delta(G) \geq (1 - 1/k)n$, where k is a divisor of n, then G has a K_k-factor.

Theorem 21 (Alon and Yuster)**.** Let H be a graph. If G satisfies

$$\delta(G) \geq \left(1 - \frac{1}{\chi(H)}\right) n,$$

then G contains at least $\big(1 - o(1)\big) \cdot n/|V(H)|$ vertex-disjoint copies of H.

5 Connectivity

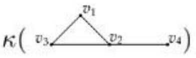
Definition.

- For a natural number $k \geq 1$, a graph G is called *k-connected* if $|V(G)| \geq k+1$ and for any set U of $k-1$ vertices in G the graph $G - U$ is connected. In particular, K_n is $(n-1)$-connected.

  k-connected

- The maximum k for which G is k-connected is called the *connectivity of* G, denoted by $\kappa(G)$.

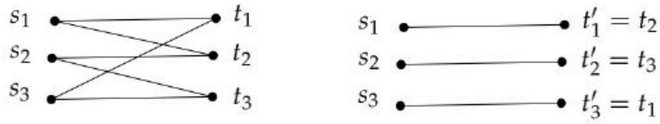
  connectivity, $\kappa(G)$

  $\kappa($ 

$) = 1,\ \kappa(C_n) = 2,\ \kappa(K_{n,m}) = \min\{m, n\}.$

- For a natural number $k \geq 1$, a graph G is called *k-linked* if for any $2k$ distinct vertices $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ there are vertex-disjoint s_i-t_i-paths, $i = 1, \ldots, k$.

  k-linked



- For a graph $G = (V, E)$ a set $X \subseteq V \cup E$ of vertices and edges of G is called a *cut set* of G if $G - X$ has more connected components than G. If a cut set consists of a single vertex v, then v is called a *cut vertex* of G; if it consists of a single edge e, then e is called a *cut edge or bridge* of G.

  cut set
  cut vertex
  cut edge, bridge

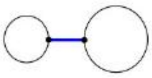
- For a natural number $\ell \geq 1$, a graph G is called *ℓ-edge-connected* if G is non-trivial and for any set $F \subseteq E$ of fewer than ℓ edges in G the graph $G - F$ is connected.

  ℓ-edge-connected

- The *edge-connectivity of* G is the maximum ℓ such that G is ℓ-edge-connected. It is denoted by $\kappa'(G)$ or $\lambda(G)$.

  edge-connectivity, $\kappa'(G)$

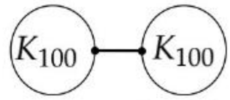
$$G \text{ non-trivial tree} \Rightarrow \lambda(G) = 1,\ G \text{ cycle} \Rightarrow \lambda(G) = 2.$$



Clearly, for every $k, \ell \geq 2$, if a graph is k-connected, k-linked or ℓ-edge-connected, then it is also $(k-1)$-connected, $(k-1)$-linked or $(\ell-1)$-edge-connected, respectively. Moreover, for a non-trivial graph is it equivalent to be 1-connected, 1-linked, 1-edge-connected, or connected.

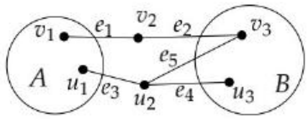
Lemma 22. For any connected, non-trivial graph G we have

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$



A graph G with $\kappa(G), \lambda(G) \ll \delta(G)$.

Definition. For a subset X of vertices and edges of G and two vertex sets A, B in G we say that X *separates* A *and* B if each A-B-path contains an element of X. separate



Some sets separating A and B: $\{e_1, e_4, e_5\}$, $\{e_1, u_2\}$, $\{u_1, u_3, v_3\}$

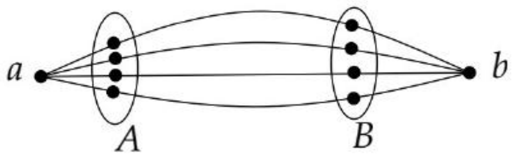
Note that if X separates A and B, then necessarily $A \cap B \subseteq X$.

Theorem 23 (Menger's Theorem, 3.3.1)**.** For any graph G and any two vertex sets $A, B \subseteq V(G)$ we have

$$\min \#\text{vertices separating } A \text{ and } B = \max \#\text{independent } A\text{-}B\text{-paths.}$$

Corollary 24. If a, b are vertices of G, $\{a, b\} \notin E(G)$, then

$$\min \#\text{vertices separating } a \text{ and } b \;=\; \max \#\text{independent } a\text{-}b\text{-paths}$$

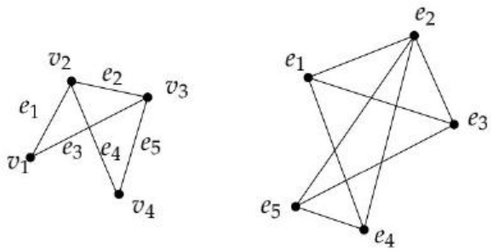


Theorem 25 (Global Version of Menger's Theorem, 3.3.6)**.** A graph G is k-connected if and only if for any two vertices a, b in G there exist k independent a-b-paths.

Note that Menger's Theorem implies that if G is k-linked, then G is k-connected. Moreover, Bollobás and Thomason proved in 1996 that if G is $22k$-connected, then G is k-linked.

Definition. For a graph $G = (V, E)$ the *line graph* $L(G)$ of G is the graph $L(G) = (E, E')$, where line graph $L(G)$

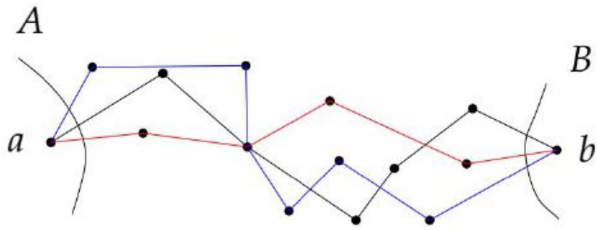
$$E' = \left\{ \{e_1, e_2\} \in \binom{E}{2} : e_1 \text{ adjacent to } e_2 \text{ in } G \right\}.$$



A graph and its line graph.

Corollary 26. If a, b are vertices of G, then

$$\min \#\text{edges separating } a \text{ and } b = \max \#\text{edge-disjoint } a\text{-}b\text{-paths}$$



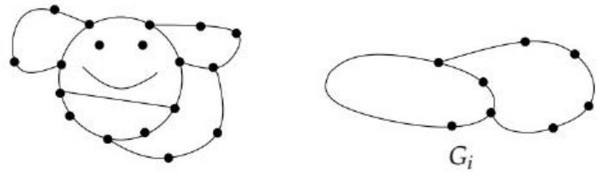
Moreover, a graph is k-edge-connected if and only if there are k edge-disjoint paths between any two vertices.

Definition. Given a graph H, we call a path P an *H-path* if P is non-trivial (has length at least one) and meets H exactly in its ends. In particular, the edge of any H-path of length 1 is never an edge of H. H-path

An *ear* of H is a non-trivial non-seperating path P in H whose internal vertices have degree 2 and whose ends have degree at least 3 each. In particular, if P is an ear of H, then P is an H'-path for the graph H' obtained from H by removing all edges and internal vertices of P. Conversely, if both ends of an H-path P lie in the same connected component of H, then P is an ear of $H + P$. ear

An *ear-decomposition* of a graph G is a sequence $G_0 \subseteq G_1 \subseteq \cdots \subseteq G_k$ of graphs, such that ear-decomposition

- G_0 is a cycle,
- for each $i = 1, \ldots, k$ the graph G_i arises from G_{i-1} by adding a G_{i-1}-path P_i, i.e., P_i is an ear of G_i, and
- $G_k = G$.

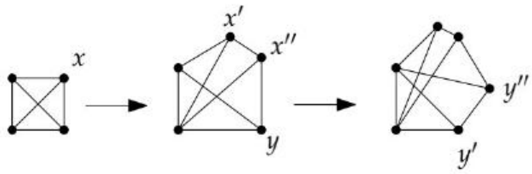


Theorem 27 (3.1.1)**.** A graph is 2-connected if and only if it has an ear-decomposition.

Lemma 28. If G is 3-connected, then there exists an edge e of G such that $G \circ e$ is also 3-connected.

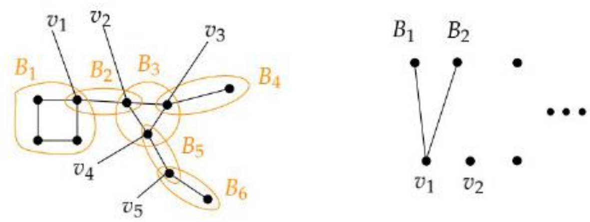
Theorem 29 (Tutte, 3.2.3)**.** A graph G is 3-connected if and only if there exists a sequence of graphs $G_0, G_1, \ldots, G_k$, such that

- $G_0 = K_4$,
- for each $i = 1, \ldots, k$ the graph G_i has two adjacent vertices x', x'' of degree at least 3, so that $G_{i-1} = G_i \circ x'x''$, and
- $G_k = G$.



Definition. Let G be a graph. A maximal connected subgraph of G without a cut vertex is called a *block* of G. In particular, the blocks of G are exactly the bridges and the maximal 2-connected subgraphs of G. block

The *block-cut-vertex graph or block graph* of G is a bipartite graph H whose partite sets are the *blocks* of G and the cut vertices of G, respectively. There is an edge between a block B and a cut vertex a if and only if $a \in B$, i.e., the block contains the cut vertex. block-cut-vertex graph



The leaves of this graph are called *block leaves*. block leaf

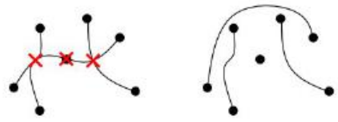
Theorem 30. The block-cut-vertex graph of a connected graph is a tree.

6 Planar graphs

This section deals with graph drawings. We restrict ourselves to graph drawings in the plane $\mathbb{R}^2$. It is also feasible to consider graph drawings in other topological spaces, such as the torus.

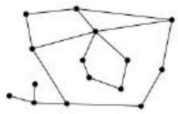
Definition.

- The *straight line segment* between $p \in \mathbb{R}^2$ and $q \in \mathbb{R}^2$ is the set $\{p+\lambda(q-p) : 0 \leq \lambda \leq 1\}$. (straight line segment)
- A *homeomorphism* is a continuous function that has a continuous inverse function. (homeomorphism)
- Two sets $A \in \mathbb{R}^2$ and $B \in \mathbb{R}^2$ are said to be *homeomorphic* if there is a homeomorphism $f\colon A \to B$. (homeomorphic)
- A *polygon* is a union of finitely many line segments that is homeomorphic to the circle $S^1 := \{x \in \mathbb{R}^2 : \|x\| = 1\}$. (polygon)
- An *arc* is a subset of $\mathbb{R}^2$ which is the union of finitely many straight line segments and is homeomorphic to the closed unit interval $[0, 1]$. The images of 0 and 1 under such a homeomorphism are the *endpoints of the arc*. If P is an arc with endpoints p and q, then P *links* them and runs *between* them. The set $P \setminus \{p, q\}$ is the *interior of* P, denoted by $\mathring{P}$. (arc; endpoint of arc; interior of arc)
- Let $O \subseteq \mathbb{R}^2$ be an open set. Being linked by an arc in O is an equivalence relation on O. The corresponding equivalence classes are the *regions of* O. A closed set $X \subseteq \mathbb{R}^2$ is said to *separate* O if $O \setminus X$ has more regions than O. The *frontier* of a set $X \subseteq \mathbb{R}^2$ is the set Y of all points $y \in \mathbb{R}^2$ such that every neighbourhood of y meets both X and $\mathbb{R}^2 \setminus X$. Note that if X is closed, its frontier lies in X, while if X is open, its frontier lies in $\mathbb{R}^2 \setminus X$. (region; separate; frontier)
- A *plane graph* is a pair (V, E) of finite sets with the following properties (the elements of V are again called *vertices*, those in E *edges*): (plane graph)
  1. $V \subseteq \mathbb{R}^2$;
  2. every $e \in E$ is an arc between two vertices;
  3. different edges have different sets of endpoints;
  4. the interior of an edge contains no vertex and no point of any other edge.

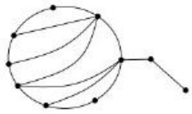


A plane graph (V, E) defines a graph G on V in a natural way. As long as no confusion can arise, we shall use the name G of this abstract graph also for the plane graph (V, E), or for the point set $V \cup \bigcup E$.

- For any plane graph G, the set $\mathbb{R}^2 \setminus G$ is open; its regions are the *faces of* G. faces, $F(G)$
- The face of G corresponding to the unbounded region is the *outer face* of G; the other faces are its *inner faces*. The set of all faces is denoted by $F(G)$. outer face inner face
- The subgraph of G whose point set is the frontier of a face f is said to *bound* f and is called its *boundary*; we denote it by $G[f]$. boundary of f, $G[f]$
- Let G be a plane graph. If one cannot add an edge to form a plane graph $G' \supsetneq G$ with $V(G') = V(G)$, then G is called *maximally plane*. If every face in $F(G)$ (including the outer face) is bounded by a triangle in G, then G is called a *plane triangulation*. maximally plane triangulation
- A *planar embedding* of an abstract graph $G = (V, E)$ is an isomorphism between G and a plane graph G'. The latter is called a *drawing* of G. We shall not distinguish notational between the vertices of G and G'. A graph $G = (V, E)$ is *planar* if it has a planar embedding. planar embedding planar graph



- A graph $G = (V, E)$ is *outerplanar* if it has a plane embedding such that the boundary of the outer face contains all of the vertices V. outerplanar graph

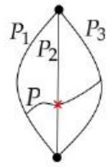


Theorem 31 (Fáry's Theorem)**.** Every planar graph has a plane embedding with straight line segments as edges.

Lemma 32 (Jordan Curve Theorem for Polygons, 4.1.1)**.** Let $P \subseteq \mathbb{R}^2$ be a polygon. Then $\mathbb{R}^2 \setminus P$ has exactly two regions. One of the regions is unbounded, the other is bounded. Each of the two regions has P as frontier.

Lemma 33. Let P_1, P_2 and P_3 be internally disjoint arcs that have the same end-points. Then

1. $\mathbb{R}^2 \setminus (P_1 \cup P_2 \cup P_3)$ has exactly three regions with boundaries $P_1 \cup P_2$, $P_1 \cup P_3$ and $P_2 \cup P_3$, respectively.
2. Let P be an arc from the interior of P_1 to the interior of P_3 whose interior lies in the region of $\mathbb{R}^2 \setminus (P_1 \cup P_3)$ containing the interior of P_2. Then P contains a points of P_2.



Lemma 34. Let G be a plane graph and e be an edge of G. Then the following hold.

- The frontier X of a face of G either contains e or is disjoint from the interior of e.
- If e is on a cycle in G, then e is on the frontier of exactly two faces.
- If e is on no cycle in G, then e is on the frontier of exactly one face.

Lemma 35. A plane graph is maximally plane if and only if it is a triangulation.

Theorem 36 (Euler's Formula, 4.2.9)**.** Let G be a connected plane graph with v vertices, e edges and f faces. Then

$$v - e + f = 2.$$

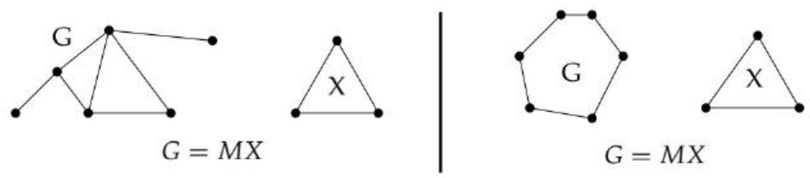
Corollary 37. Let $G = (V, E)$ be a plane graph. Then

- $|E| \leq 3|V| - 6$ with equality exactly if G is a plane triangulation.
- $|E| \leq 2|V| - 4$ if no face in $F(G)$ is bounded by a triangle.

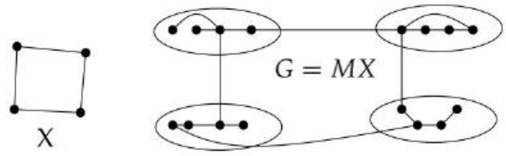
Lemma 38 (Pick's Formula)**.** Let P be a polygon with corners on the grid $\mathbb{Z}^2$, let A be its area, I be the number of grid points strictly inside of P and B be the number of grid points on the boundary of P. Then $A = I + B/2 - 1$.

Definition. Let G and X be two graphs.

- We say that G *is an* MX, denoted by $G = MX$, if $V(G)$ can be partitioned as $\{V_x \mid x \in V(X)\}$ such that $G[V_x]$ is connected for every $x \in V(X)$ and there is an V_x–V_y edge in G if and only if $xy \in E(X)$. MX, $G = MX$
- We say that X *is a minor of* G if $H = MX$ for some subgraph H of G. minor, $X \preccurlyeq G$



Alternatively, X is a minor of G if and only if X can be obtained from G by successive vertex deletions, edge deletions and edge contractions.



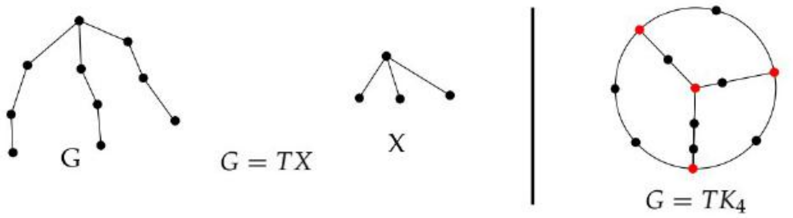
- The graph G is a *single-edge subdivision of* X if $V(G) = V(X) \cup \{v\}$ and $E(G) = E(X) - xy + xv + vy$ for some edge $xy \in E(X)$ and $v \notin V(X)$. We say that G *is a* TX, denoted by $G = TX$, if G can be obtained from X by a series of single-edge subdivisions.

  subdivision

  TX, $G = TX$

- We say that X *is a topological minor of* G, if $H = TX$ for some subgraph H of G.

  topological minor



Theorem 39 (Kuratowski's Theorem, 4.4.6)**.** A graph is planar if and only if it does not have K_5 or $K_{3,3}$ as topological minors.

Definition.

- Let X be a set and $\leq \subseteq X^2$ be a relation on X, i.e., $\leq$ is a subset of all ordered pairs of elements in X. Then $\leq$ is a *partial order* if it is reflexive, antisymmetric and transitive. A partial order is *total* if $x \leq y$ or $y \leq x$ for every $x, y \in X$.

  partial order
  total order

- Let $\leq$ be a partial order on a set X. The pair $(X, \leq)$ is called a *poset* (partially ordered set). If $\leq$ is clear from context, the set X itself is called a poset. The *poset dimension of* $(X, \leq)$ is the smallest number d such that there are total orders $R_1, \ldots, R_d$ on X with $\leq = R_1 \cap \cdots \cap R_d$.

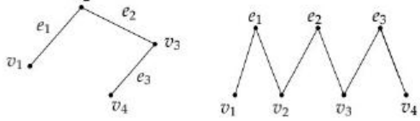
  poset

  poset dimension, $\dim(X, \leq)$

  $$\dim(\vdots) = 1,\ \dim(\underset{x}{\bullet}\ \underset{y}{\bullet}) = 2 \text{ since } \underset{x}{\bullet}\ \underset{y}{\bullet} = \overset{x}{\underset{y}{\vert}} \cap \overset{y}{\underset{x}{\vert}}$$

- The *incidence poset* $(V \cup E, \leq)$ on a graph $G = (V, E)$ is given by $v \leq e$ if and only if e is incident to v for all $v \in V$ and $e \in E$.

  incidence poset



Theorem 40 (Schnyder)**.** Let G be a graph and P be its incidence poset. Then G is planar if and only if $\dim(P) \leq 3$.

Theorem 41 (5-Color Theorem, 5.1.2)**.** Every planar graph is 5-colorable.

The more well-known 4-coloring theorem is much harder to prove. Interestingly, it is one of the first theorems that has been proved using computer assistance. The computer-generated proof uses an enormous case distinction. Some mathematicians have philosophical problems with this approach since the resulting proof cannot be easily verified by humans. A shorter proof is still outstanding.

Theorem 42 (4-Color Theorem, 5.1.1)**.** Every planar graph is 4-colorable.

Definition.

- Let $L(v) \subseteq \mathbb{N}$ be a list of colors for each vertex $v \in V$. We say that G is *L-list-colorable* if there is coloring $c\colon V \to \mathbb{N}$ such that $c(v) \in L(v)$ for each $v \in V$ and adjacent vertices receive different colors.
- Let $k \in \mathbb{N}$. We say that G is *k-list-colorable or k-choosable* if G is L-list-colorable for each list L with $|L(v)| = k$ for all $v \in V$.
- The *choosability*, denoted by $\text{ch}(G)$, is the smallest k such that G is k-choosable.
- The *edge choosability*, denoted by $\text{ch}'(G)$, is defined analogously.

L-list-colorable

k-list-colorable

choosability, $\text{ch}(G)$

edge choosability, $\text{ch}'(G)$

Theorem 43 (Thomassen's 5-List Color Theorem, 5.4.2)**.** Every planar graph is 5-choosable.

7 Colorings

Lemma 44 (Greedy estimate for the chromatic number).
Let G be a graph. Then $\chi(G) \leq \Delta(G) + 1$.

Theorem 45 (Brook's Theorem, 5.2.4). Let G be a connected graph. Then $\chi(G) \leq \Delta(G)$ unless G is a complete graph or an odd cycle.

Definition.

- The *clique number* $\omega(G)$ *of* G is the largest order of a clique in G. clique number, $\omega(G)$
- The *co-clique number* $\alpha(G)$ *of* G is the largest order of an independent set in G. co-clique number, $\alpha(G)$
- A graph G is called *perfect* if $\chi(H) = \omega(H)$ for each induced subgraph H of G. For example, bipartite graphs are perfect with $\chi = \omega = 2$. perfect graph

Lemma 46 (Small Coloring Results).

- $\chi(G) \geq \max\{\omega(G), n/\alpha(G)\}$ since each color class is an empty induced subgraph and $\chi(K_k) = k$.
- $\|G\| \geq \binom{\chi(G)}{2} \Leftrightarrow \chi(G) \leq 1/2 + \sqrt{2\|G\| + 1/4}$ since there must be at least one edge between any two color classes.
- The chromatic number $\chi(G)$ of G is at most one more than the length of a longest directed path in any orientation of G.

Theorem 47 (Lovász' Perfect Graph Theorem, 5.5.4). A graph G is perfect if and only if its complement $\overline{G}$ is perfect.

Theorem 48 (Strong Perfect Graph Theorem, Chudnovsky, Robertson, Seymour & Thomas, 5.5.3). A graph G is perfect if and only if it does not contain an odd cycle on at least 5 vertices (an *odd hole*) or the complement of an odd hole as an induced subgraph.

Definition. For an integer $k \geq 1$ we define k-*constructible* graphs recursively as follows: k-constructible

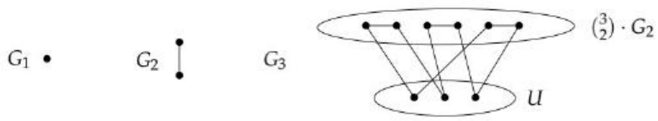
- K_k is k-constructible.
- If G is k-constructible and $x, y \in V(G)$ are non-adjacent, then also $(G + xy)/xy$ is k-constructible.
- If G_1, G_2 are k-constructible and there are vertices x, y_1, y_2 such that $G_1 \cap G_2 = \{x\}$, $xy_1 \in E(G_1)$ and $xy_2 \in E(G_2)$, then also $(G_1 \cup G_2) - xy_1 - xy_2 + y_1y_2$ is k-constructible.

Theorem 49 (Hajós Theorem, 5.2.6). Let G be a graph and $k \geq 1$ be an integer. Then $\chi(G) \geq k$ if and only if G has a k-constructible subgraph.

Example (Mycielski's Construction).
We can construct a family $\big(G_k = (V_k, E_k)\big)_{k\in\mathbb{N}}$ of triangle-free graphs with $\chi(G_k) = k$ as follows:

- G_1 is the single-vertex graph, G_2 is the single-edge graph, i.e., $G_1 = K_1$ and $G_2 = K_2$.
- $V_{k+1} := V_k \cup U \cup \{w\}$ where $V_k \cap \big(U \cup \{w\}\big) = \emptyset$, $V_k = \{v_1, \dots, v_n\}$ and $U = \{u_1, \dots, u_n\}$.
- $E_{k+1} := E_k \cup \big\{wu_i : i = 1, \dots, k\big\} \cup \bigcup_{i=1}^{n} \big\{u_i v : v \in N_{G_k}(v_i)\big\}$.

Example (Tutte's Construction). We can construct a family $(G_k)_{k\in\mathbb{N}}$ of triangle-free graphs with $\chi(G_k) = k$ as follows: G_1 is the single-vertex graph. To get from G_k to G_{k+1}, take an independent set U of size $k(|G_k| - 1) + 1$ and $\binom{|U|}{|G_k|}$ vertex-disjoint copies of G_k. For each subset of size $|G_k|$ in U then introduce a perfect matching to exactly one of the copies of G_k.



Theorem 50 (König's Theorem, 5.3.1).
Let G be a bipartite graph. Then $\chi'(G) = \Delta(G)$.

Theorem 51 (Vizing's Theorem, 5.3.2).
Let G be a graph. Then $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$.

Lemma 52. We have $\mathrm{ch}(K_{n,n}) \geq c \cdot \log(n)$ for some constant $c > 0$. In particular,

$$\mathrm{ch}\left(K_{\binom{2k-1}{k},\binom{2k-1}{k}}\right) \geq c \cdot k.$$

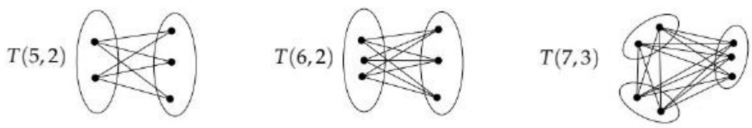
Theorem 53 (Galvin's Theorem, 5.4.4).
Let G be a bipartite graph. Then $\mathrm{ch}'(G) = \chi'(G)$.

8 Extremal graph theory

In this section $c, c_1, c_2, \dots$ always denote unspecified constants in $\mathbb{R}_{>0}$.

Definition.

- Let n be a positive integer and H a graph. By $\mathrm{ex}(n, H)$ we denote the maximum size of a graph of order n that does not contain H as a subgraph; $\mathrm{EX}(n, H)$ is the set of such graphs.
- Let n and r be integers with $1 \leq r \leq n$. The *Turán graph* $T(n,r)$ is the unique complete r-partite graph of order n whose partite sets differ by at most 1 in size. It does not contain K_{r+1}. We denote $\|T(n,r)\|$ by $t(n,r)$.



- In the special case that $n = r \cdot s$, for positive integers n, r, s with $1 \leq r \leq n$, the Turán graph $T(n,r)$ is also denoted by K_r^s.

$\mathrm{ex}(n, H)$

$\mathrm{EX}(n, H)$

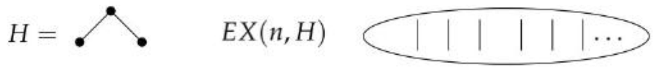
Turán graph, $T(n,r)$

$t(n,r)$

K_r^s

Example.

- $\mathrm{ex}(n, K_2) = 0$, $\mathrm{EX}(n, K_2) = \{E_n\}$
- $\mathrm{ex}(n, P_3) = \lfloor n/2 \rfloor$, $\mathrm{EX}(n, P_3) = \{\lfloor n \rfloor \cdot K_2 + (n \bmod 2) \cdot E_1\}$



Lemma 54 (On Turán Graphs)**.**

- Among all r-partite graphs on n vertices the Turán graph $T(n,r)$ has the largest number of edges.
- We have the recursion

$$t(n,r) = t(n-r, r) + (n-r)(r-1) + \binom{r}{2}.$$

- A Turán graph lacks a ratio of $1/r$ of the edges of a complete graph:

$$\lim_{n \to \infty} \frac{t(n,r)}{\binom{n}{2}} = \left(1 - \frac{1}{r}\right).$$

Theorem 55 (Turán's Theorem, 7.1.1)**.** For all integers $r > 1$ and $n \geq 1$, any graph G with n vertices, $\mathrm{ex}(n, K_r)$ edges and $K_r \not\subseteq G$ is a $T_{r-1}(n)$.

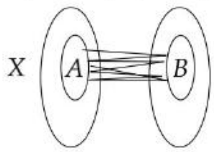
In other words $\mathrm{EX}(n, K_r) = \{T(n, r-1)\}$.

Definition. Let $X, Y \subseteq V(G)$ be disjoint vertex sets and $\epsilon > 0$.

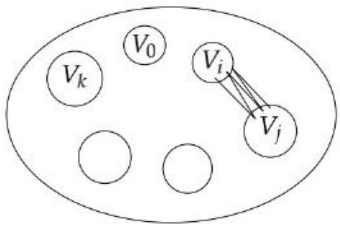
- We define $\|X, Y\|$ to be the number of edges between X and Y and *the density* $d(X,Y)$ *of* (X, Y) to be density, $d(X,Y)$

$$d(X,Y) := \frac{\|X,Y\|}{|X||Y|}.$$

- For $\epsilon > 0$ the pair (X, Y) is an *ϵ-regular pair* if we have $|d(X,Y) - d(A,B)| \leq \epsilon$ for all $A \subseteq X$, $B \subseteq Y$ with $|A| \geq \epsilon|X|$ and $|B| \geq \epsilon|Y|$. ϵ-regular pair



- An *ϵ-regular partition* of the graph $G = (V, E)$ is a partition of the vertex set $V = V_0 \dot\cup V_1 \dot\cup \cdots \dot\cup V_k$ with the following properties: ϵ-regular partition
  1. $|V_0| \leq \epsilon|V|$
  2. $|V_1| = |V_2| = \cdots = |V_k|$
  3. All but at most ϵk^2 of the pairs (V_i, V_j) for $1 \leq i < j \leq k$ are ϵ-regular.



Theorem 56 (Szemerédi's Regularity Lemma, 7.4.1)**.** For any $\epsilon > 0$ and any integer $m \geq 1$ there is an $M \in \mathbb{N}$ such that every graph of order at least m has an ϵ-regular partition $V_0 \dot\cup \cdots \dot\cup V_k$ with $m \leq k \leq M$.

Theorem 57 (Erdős-Stone Theorem, 7.1.2)**.** For all integers $r > s \geq 1$ and any $\epsilon > 0$ there exists an integer n_0 such that every graph with $n \geq n_0$ vertices and at least

$$t_{r-1}(n) + \epsilon n^2$$

edges contains K_r^s as a subgraph.

Corollary 58. Erdős-Stone together with $\lim_{n\to\infty} t(n,r)/\binom{n}{2} = 1 - 1/r$ yields an asymptotic formula for the extremal number of any graph H on at least one edge:

$$\lim_{n\to\infty} \frac{\mathrm{ex}(n,H)}{\binom{n}{2}} = \frac{\chi(H)-2}{\chi(H)-1}$$

For example, $\mathrm{ex}(n,$  $) \simeq 2/3 \cdot \binom{n}{2}$ since $\chi($  $) = 4$.

Chvátal and Szemerédi proved a more quantitative version of the Erdős-Stone theorem.

Theorem 59 (Chvátal-Szemerédi Theorem)**.** For any $\epsilon > 0$ and any integer $r \geq 3$, any graph on n vertices and at least $\left(1 - 1/(r-1) + \epsilon\right)\binom{n}{2}$ edges contains K_r^t as a subgraph. Here t is given by

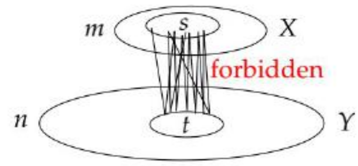
$$t = \frac{\log n}{500 \cdot \log(1/\epsilon)}.$$

Furthermore, there is a graph G on n vertices and $\left(1 - (1+\epsilon)/(r-1)\right)\binom{n}{2}$ edges that does not contain K_r^t for

$$t = \frac{5 \cdot \log n}{\log(1/\epsilon)},$$

i.e., the choice of t is asymptotically tight.

Definition. The *Zarankiewicz* function $z(m,n;s,t)$ denotes the maximum number of edges that a bipartite graph with parts of size m and n can have without containing $K_{s,t}$.



Zarankiewicz, $z(m,n;s,t)$

Theorem 60 (Kővári-Sós-Turán Theorem)**.**
We have the upper bound

$$z(m,n;s,t) \leq (s-1)^{1/t}(n-t+1)m^{1-1/t} + (t-1)m$$

for the Zarankiewicz function. In particular,

$$z(n,n;t,t) \leq c_1 \cdot n \cdot n^{1-1/t} + c_2 \cdot n = \mathcal{O}(n^{2-1/t})$$

for $m = n$ and $t = s$.

Corollary 61.
For $t \geq s \geq 1$ we can bound the extremal number of $K_{t,s}$ using the Kővári-Sós-Turán theorem

$$\mathrm{ex}(n, K_{t,s}) \leq \frac{1}{2} \cdot z(n,n;s,t) \leq cn^{2-1/s}.$$

For $t = s = 2$ this bound yields

$$\mathrm{ex}(n, C_4) \leq \frac{n}{4}(1 + \sqrt{4n-3}).$$

This bound is actually tight, i.e., $\mathrm{ex}(n, C_4) = 1/2 \cdot n^{3/2} \cdot (1 + o(1))$.

Lemma 62. $\mathrm{ex}(n, K_{r,r}) \geq cn^{2-2/(r+1)}$ for all $n, r \in \mathbb{N}$.

Theorem 63. For all $n \in \mathbb{N}$ we have $\mathrm{ex}(n, P_{k+1}) \leq \big(n \cdot (k-1)\big)/2$.

Conjecture (Hadwiger Conjecture)**.** Let r be a natural number and G be a graph. Then $\chi(G) \geq r$ implies $MK_r \subseteq G$.

For $r \in \{1, 2, 3, 4\}$ this is easy to see. For $r \in \{5, 6\}$ the conjecture has been proven using the 4-color-theorem. It is still open for $r \geq 7$.

Theorem 64 (Bollobás-Thomason 1998, 7.2.1)**.** Every graph G of average degree at least cr^2 contains K_r as a topological minor.

Theorem 65 (7.2.4)**.** Let G be a graph of minimum degree $\delta(G) \geq d$ and girth $g(G) \geq 8k + 3$ for $d, k \in \mathbb{N}$ and $d \leq 3$. Then G has a minor H of minimum degree $\delta(H) \geq d(d-1)^k$.

Theorem 66 (Thomassen's Theorem, 7.2.5)**.** For all $r \in \mathbb{N}$ there exists a function $f\colon \mathbb{N} \to \mathbb{N}$ such that every graph of minimum degree at least 3 and girth at least $f(r)$ has a K_r minor.

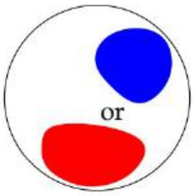
Theorem 67 (Kühn-Osthus, 7.2.6)**.** Let $r \in \mathbb{N}$. Then there is a constant $g \in \mathbb{N}$ such that we have $TK_r \subseteq G$ for every graph G with $\delta(G) \geq r - 1$ and $g(G) \geq g$.

9 Ramsey theory

In every 2-coloring in this section we use the colors red and blue.

Definition.

- In an edge-coloring of a graph, a set of edges is
  - *monochromatic* if all edges have the same color, monochromatic
  - *rainbow* if no two edges have the same color, rainbow
  - *lexical* if two edges have the same color if and only if they have the same lower endpoint in some ordering of the vertices. lexical
- Let k be a natural number. Then the *Ramsey number* $R(k) \in \mathbb{N}$ is the smallest n such that every 2-edge-coloring of K_n contains a monochromatic K_k. Ramsey, $R(k)$

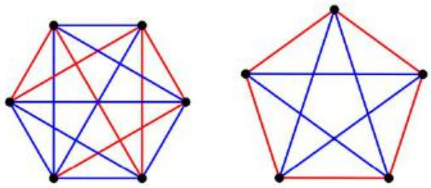


Color $E(K_n)$ in 2 colors.

- Let k and l be natural numbers. Then the *asymmetric Ramsey number* $R(k, l)$ is the smallest $n \in \mathbb{N}$ such that every 2-edge-coloring of a K_n contains a red K_k or a blue K_l. asymmetric Ramsey, $R(k, l)$
- Let G and H be graphs. Then the *graph Ramsey number* $R(G, H)$ is the smallest $n \in \mathbb{N}$ such that every 2-edge-coloring of K_n contains a red G or a blue H. graph Ramsey, $R(G, H)$
- Let $r, l_1, \dots, l_k$ be natural numbers. Then the *hypergraph Ramsey number* $R_r(l_1, \dots, l_k)$ is the smallest $n \in \mathbb{N}$ such that for every k-coloring of $\binom{[n]}{r}$ there is an $i \in \{1, \dots, k\}$ and a $V \subseteq [n]$ with $|V| = l_i$ such that all sets in $\binom{V}{r}$ have color i. hypergraph Ramsey, $R_r(l_1, \dots, l_k)$
- Let G and H be graphs. Then the *induced Ramsey number* $R_{ind}(G, H)$ is the smallest $n \in \mathbb{N}$ such that there is a graph F on n vertices every 2-coloring of which contains a red G or a blue H. induced Ramsey, $R_{\text{ind}}(G, H)$
- For $n \in \mathbb{N}$ and a graph H, the *anti-Ramsey number* $AR(n, H)$ is the maximum number of colors that an edge-coloring of K_n can have without containing a rainbow copy of H. anti-Ramsey, $AR(n, H)$

Lemma 68.

- $R(3) = 6$, i.e., every 2-edge-colored K_6 contains a monochromatic K_3 and there is a 2-coloring of a K_5 without monochromatic K_3's.



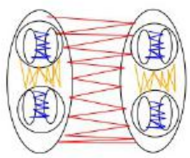
- Clearly, $R(2,k) = R(k,2) = k$.

Theorem 69 (Ramsey Theorem, 9.1.1)**.** For any $k \in \mathbb{N}$ we have $\sqrt{2}^k \leq R(k) \leq 4^k$. In particular, the Ramsey numbers, the asymmetric Ramsey numbers and the graph Ramsey numbers are finite.

Theorem 70. For any $k, l \in \mathbb{N}$ we have $R(k,l) \leq R(k-1,l) + R(k,l-1)$. This implies $R(k,l) \leq \binom{k+l-2}{k-1}$ by induction.

Lemma 71. For any $r, p, q \in \mathbb{N}$ we have $R_r(p,q) \leq R_{r-1}\big(R_r(p-1,q), R_r(p,q-1)\big) + 1$.

Lemma 72. We have $c_1 \cdot 2^k \leq R_2(\underbrace{3,\dots,3}_{k}) \leq c_2 \cdot k!$ for some constants $c_1, c_2 > 0$.



Applications of Ramsey theory

Theorem 73 (Erdős-Szekeres)**.** Any sequence of $(r-1)(s-1)+1$ distinct real numbers contains an increasing subsequence of length r or a decreasing subsequence of length s.

Theorem 74 (Erdős-Szekeres)**.** For any $m \in \mathbb{N}$ there is an $N \in \mathbb{N}$ such that every set of at least N points in general position in $\mathbb{R}^2$ contains the vertex set of a convex m-gon.

Theorem 75 (Schur)**.** Let $c\colon \mathbb{N} \to [r]$ be a coloring of the natural numbers with $r \in \mathbb{N}$ colors. Then there are $x, y, z \in \mathbb{N}$ of the same color with $x + y = z$.

Definition. Let $r \in \mathbb{N}$ and $A \in \mathbb{Z}^{n \times k}$.

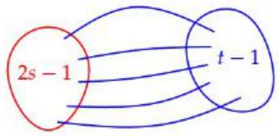
- Matrix A is said to be *r-regular* if there is a monochromatic solution of $Ax = 0$ for any r-coloring $c\colon \mathbb{N} \to [r]$ of $\mathbb{N}$. r-regular matrix
- Matrix A fulfils the *column condition* if there is a partition $C_1 \dot\cup \cdots \dot\cup C_l$ of the columns of A such that the following holds: Let $s_i := \sum_{c \in C_i} c$ for $i \in [l]$ be the sum of columns in C_i. Then $s_1 = 0$ and every s_i for $i \in \{2, \dots, l\}$ is a linear combination of the columns in $C_1 \dot\cup \dots \dot\cup C_{i-1}$. column condition

  For example, $2x_1 + x_2 + x_3 - 4x_4$ fulfils the column condition since $2 + 1 +$

$1 - 4 = 0.$

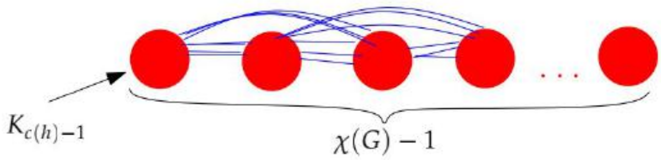
Theorem 76 (Rado)**.** Let $A \in \mathbb{Z}^{n\times k}$. If A fulfils the column condition, then A is r-regular for every $r \in \mathbb{N}$.

Lemma 77. For any $s, t \in \mathbb{N}$ with $s \geq t \geq 1$ we have $R(sK_2, tK_2) = 2s + t - 1$.



Lemma 78. For any $s, t \in \mathbb{N}$ with $s \geq t \geq 1$ and $s \geq 2$ we have $R(sK_3, tK_3) = 3s+2t$.

Theorem 79 (Chvátal, Harary)**.** Let G and H be graphs. Then $R(G, H) \geq \big(\chi(G) - 1\big)\big(c(H) - 1\big) + 1$ where $c(H)$ is the order of the largest component of H.



Theorem 80 (Induced Ramsey Theorem, Deuber, Erdős, Hajnal & Pósa, 9.3.1)**.** We have that $R_{\text{ind}}(G, H)$ is finite for all graphs G and H.

Theorem 81 (Canonical Ramsey Theorem, Erdős-Rado 1950)**.** For all $k \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that any edge coloring of K_n with arbitrarily many colors contains a K_k that is monochromatic, rainbow or lexical.

Theorem 82 (Chvátal-Rödl-Szemerédi-Trotter, 9.2.2)**.** For any positive integer Δ there exists a $c \in \mathbb{N}$ such that for every graph H with $\Delta(H) = \Delta$ we have $R(H, H) \leq c|V(H)|$.

Corollary 83. For any n-vertex graph H with maximum degree 3 we have $R(H, H) \leq cn$ for some constant $c > 0$. This number grows much slower than $R(K_n, K_n) \geq \sqrt{2}^n$.

Theorem 84 (Anti-Ramsey Theorem, Erdős-Simonvits-Sós)**.** For all $n, r \in \mathbb{N}$ we have $AR(n, K_r) = \binom{n}{2}\big(1 - 1/(r-2)\big)\big(1 - o(1)\big)$.

10 Flows

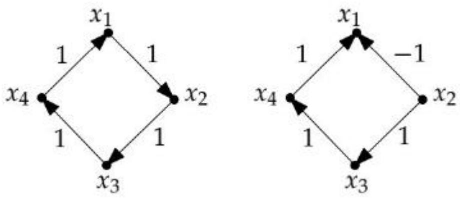
Definition. Let H be an Abelian semigroup, let $G = (V, E)$ be a multigraph and let $\tilde{E} := \{(x, y) : xy \in E\}$.

- For $f : \tilde{E} \to H$ and $X, Y \subseteq V$ we define $f(X, Y) := \sum_{(x,y) \in (X \times Y) \cap \tilde{E}} f(x, y)$.
- A function $f : \tilde{E} \to H$ is a *circulation on G* if

  circulation

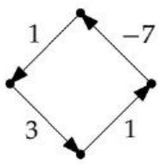
  (C_1) $f(x, y) = -f(y, x)$ for all $xy \in E$ and

  (C_2) $f(v, V) = 0$ for all $v \in V$.



- If H is an Abelian group, then a circulation f is also called an *H-flow on G*. If $f(x, y) \neq 0$ for all $xy \in E$, then f is a *nowhere-zero flow*.

  H-flow
  nowhere-zero



A nowhere-zero $\mathbb{Z}_2$-flow.

- For $k \in \mathbb{N}$ a *k-flow* is a $\mathbb{Z}$-flow f such that $0 < |f(x, y)| < k$ for all $xy \in E$. The *flow number* $\varphi(G)$ *of G* is the smallest k such that G has a k-flow.

  k-flow
  flow number, $\varphi(G)$

- Let $s \in V$ and $t \in V$ be two distinct vertices, $c : \tilde{E} \to \mathbb{Z}_{\geq 0}$ be a function on $\tilde{E}$ with non-negative integer values. Then the tuple (G, s, t, c) is called a *network* with *source s*, *sink t* and *capacity function c*.

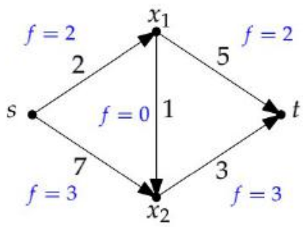
  network, source, sink, capacity, network flow

- A *network flow* is a function $f : \tilde{E} \to \mathbb{R}$ with the following properties for all $x, y \in V$:

  (F_1) $f(x, y) = -f(y, x)$

  (F_2) $f(x, V) = 0$ if $x \notin \{s, t\}$

  (F_3) $f(x, y) \leq c(x, y)$



For any $S \subseteq V$ with $s \in S$ and $t \notin S$ the pair $(S, V \setminus S)$ is called a *cut*. Its capacity is $c(S, V \setminus S)$. cut

The value $f(s, V)$ is also called the *value of* f and is denoted by $|f|$. value, $|f|$

Lemma 85.

- For any circulation f and $X \subseteq V$ we have $f(X, X) = 0$, $f(X, V) = 0$ and $f(X, V \setminus X) = 0$
- For any network flow f and cut $(S, \bar{S})$ we have $f(S, \bar{S}) = f(s, V)$.

Theorem 86 (Ford-Fulkerson Theorem, 6.2.2)**.** In any network the maximum value of a flow is the same as the minimum capacity of a cut and there is an integral flow $f\colon \tilde{E} \to \mathbb{Z}_{\geq 0}$ with this maximum flow value.

Theorem 87 (Tutte, 6.3.1)**.** For every multigraph G there is a polynomial $P \in \mathbb{Z}[X]$ such that for any finite Abelian group H the number of nowhere-zero H-flows on G is $P\big(|H| - 1\big)$.

Corollary 88. If an H-flow on G exists for some finite Abelian group H, then there is also an $\tilde{H}$-flow on G for all finite Abelian groups $\tilde{H}$ with $|\tilde{H}| = |H|$. For example, if a $\mathbb{Z}_4$-flow exists, then a $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow also exists.

Theorem 89 (Tutte, 6.3.3)**.** A multigraph admits a k-flow if and only if it admits a $\mathbb{Z}_k$-flow.

Theorem 90 (Tutte, 6.5.3)**.** For a planar graph G and its dual G^* we have $\chi(G) = \varphi(G^*)$.

Lemma 91. A graph has a 2-flow if and only if all of its degrees are even.

Lemma 92. A cubic (3-regular) graph has a 3-flow if and only if it is bipartite.

Conjecture (Tutte's 5-Flow Conjecture)**.** Every bridgeless multigraph has flow number at most 5.

Theorem 93 (Seymour, 6.6.1)**.** Every bridgeless graph has flow number at most 6.

11 Random graphs

In this section we deal with randomly chosen graphs. We will often use the "probabilistic method", a proof method for showing existence: By proving that an object with some desired properties can be chosen randomly (in some probability space) with non-zero probability, we also show that such an object exists.

Definition.

- $\mathcal{G}(n,p)$ is the probability space on all n-vertex graphs that results from independently deciding whether to include each of the $\binom{n}{2}$ possible edges with fixed probability $p \in [0,1]$. This model is called the *Erdős-Rényi model* of random graphs. Erdős-Rényi
- A *property* $\mathcal{P}$ is a set of graphs, e.g. $\mathcal{P} = \{G : G \text{ is } k\text{-connected}\}$. property

  Let $(p_n) \in [0,1]^{\mathbb{N}}$ be a sequence. We say that $G \in \mathcal{G}(n,p_n)$ *almost always* has property $\mathcal{P}$ if $\text{Prob}(G \in \mathcal{G}(n,p_n) \cap \mathcal{P}) \to 1$ for $n \to \infty$. If (p_n) is constant p, we also say in this case that *almost all* graphs in $\mathcal{G}(n,p)$ have property $\mathcal{P}$. almost always
- A function $f(n)\colon \mathbb{N} \to [0,1]$ is a *threshold function* for property $\mathcal{P}$ if: threshold function
  - For all $(p_n) \in [0,1]^{\mathbb{N}}$ with $p_n/f(n) \overset{n\to\infty}{\longrightarrow} 0$ the graph $G \in \mathcal{G}(n,p_n)$ almost always does not have property $\mathcal{P}$.
  - For all $(p_n) \in [0,1]^{\mathbb{N}}$ with $p_n/f(n) \overset{n\to\infty}{\longrightarrow} \infty$ the graph $G \in \mathcal{G}(n,p_n)$ almost always has property $\mathcal{P}$.

  Note that not all properties $\mathcal{P}$ have a threshold function.

Lemma 94.

- For a given graph G on n vertices and m edges we have

$$\text{Prob}\big(G = \mathcal{G}(n,p)\big) = p^m(1-p)^{\binom{n}{2}-m}.$$

- For all integers $n \geq k \geq 2$ we have

$$\text{Prob}\big(G \in \mathcal{G}(n,p), \alpha(G) \geq k\big) \leq \binom{n}{k}(1-p)^{\binom{k}{2}}$$

and

$$\text{Prob}\big(G \in \mathcal{G}(n,p), \omega(G) \geq k\big) \leq \binom{n}{k}p^{\binom{k}{2}}.$$

Theorem 95 (Erdős, 11.1.3). Erdős proved the lower bound $R(k,k) \geq 2^{k/2}$ on Ramsey numbers by applying the probabilistic method to the Erdős-Rényi model.

Lemma 96 (11.1.5). We have

$$\mathrm{Exp}\big(\#k\text{-cycles in } G \in \mathcal{G}(n,p)\big) = \frac{n_k}{2k} \cdot p^k$$

where $n_k = n \cdot (n-1) \cdots (n-k+1)$.

Theorem 97 (Erdős, 11.2.2). For any $k \in \mathbb{N}$ there is a graph H with $g(H) \geq k$ and $\chi(H) \geq k$.

Lemma 98 (11.3.1). For all $p \in (0,1)$ and any graph H almost all graphs in $\mathcal{G}(n,p)$ contain H as an induced subgraph.

Lemma 99 (11.3.4). For all $p \in (0,1)$ and $\epsilon > 0$ almost all graphs G in $\mathcal{G}(n,p)$ fulfil

$$\chi(G) > \frac{\log\big(1/(1-p)\big)}{2+\epsilon} \cdot \frac{n}{\log n}.$$

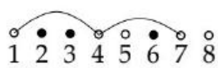
Remark. Asymptotic behaviour of $\mathcal{G}(n,p)$ for some properties:

- $p_n = \sqrt{2}/n^2 \Rightarrow G$ almost always has a component with > 2 vertices
- $p_n = 1/n \Rightarrow G$ almost always has a cycle
- $p_n = \log n/n \Rightarrow G$ is almost always connected
- $p_n = (1+\epsilon)\log n/n \Rightarrow G$ almost always has a Hamiltonian cycle
- $p_n = n^{-2/(k-1)}$ is the threshold function for containing K_k

Lemma 100 (Lovász Local Lemma). Let $A_1, \ldots, A_n$ be events in some probabilistic space. If $\mathrm{Prob}(A_i) \leq p \in (0,1)$, each A_i is mutually independent from all but at most $d \in \mathbb{N}$ A_is and $ep(d+1) \leq 1$, then

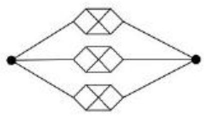
$$\mathrm{Prob}\left(\bigwedge_{i=1}^{n} \overline{A_i}\right) > 0.$$

Lemma 101. *Van-der-Waerden's number* $W(k)$ is the smallest n such that any 2-coloring of $[n]$ contains a monochromatic arithmetic progression of length k. We can prove $W(k) \geq 2^{k-1}/(ek^2)$ with the Lovász Local Lemma.



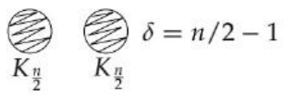
12 Hamiltonian cycles

Lemma 102 (Necessary condition for the existence of a Hamiltonian cycle)**.** If G has a Hamiltonian cycle, then for every non-empty $S \subseteq V$ the graph $G - S$ cannot have more than $|S|$ components.



Non-hamiltonian graph.

Theorem 103 (Dirac, 10.1.1)**.** Every graph with $n \geq 3$ vertices and minimum degree at least $n/2$ has a Hamiltonian cycle.

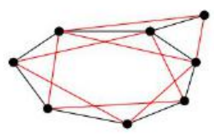


Theorem 104. Every graph on $n \geq 3$ vertices with $\alpha(G) \leq \kappa(G)$ is Hamiltonian.

Theorem 105 (Tutte, 10.1.4)**.** Every 4-connected planar graph is Hamiltonian.

Definition. Let $G = (V, E)$ be a graph. The *square* of G, denoted by G^2, is the graph $G^2 := (V, E')$ with $E' := \{uv : u, v \in V, d_G(u, v) \leq 2\}$. square, G^2

Theorem 106 (Fleischner's Theorem, 10.3.1)**.** If G is 2-connected, then G^2 is Hamiltonian.



Theorem 107 (Chvátal, 10.2.1)**.** Let $0 \leq a_1 \leq \cdots \leq a_n < n$ be an integer sequence with $n \geq 3$. A graph with the degree sequence $a_1, \ldots, a_n$ is Hamiltonian if and only if $a_i \leq i$ implies $a_{n-i} \geq n - i$ for all $i < n/2$.

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