SUBJECT: MATHEMATICS

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Solve the following Fredholm integral equ^{n}.

(1) $\phi(x) = \sec^2 x + \lambda \int_0^1 \phi(\xi)\, d\xi$

Solution. Given $\phi(x) = \sec^2 x + \lambda \int_0^1 \phi(\xi)\, d\xi$ ——— (1)

We know that solu^{n} of

$$\phi(x) = f(x) + \lambda \int_a^b K(x,\xi)\phi(\xi)\, d\xi \quad \text{is}$$

$$\phi(x) = f(x) + \lambda \int_a^b R(x,\xi;\lambda) f(\xi)\, d\xi$$

Here $f(x) = \sec^2 x$, $K(x,\xi) = 1$, $a = 0$, $b = 1$

We know that

$$R(x,\xi;\lambda) = \frac{D(x,\xi;\lambda)}{D(\lambda)}$$

where $D(x,\xi;\lambda) = B_0(x,\xi) - \frac{\lambda}{\underline{|1}} B_1(x,\xi) + \frac{\lambda^2}{\underline{|2}} B_2(x,\xi) - - -$

$$D(\lambda) = 1 - \frac{\lambda}{\underline{|1}} C_1 + \frac{\lambda^2}{\underline{|2}} C_2 - \frac{\lambda^3}{\underline{|3}} C_3 + - -$$

Here $B_0(x,\xi) = 1$, $C_1 = \int_0^1 B_0(\xi_1,\xi_1)\, d\xi_1 = \int_0^1 1 . d\xi_1$

$$\Rightarrow \boxed{C_1 = 1}$$

We know that

$$B_m(x,\xi) = C_m K(x,\xi) - m \int_0^1 K(x,\xi_1) B_{m-1}(\xi_1,\xi)\, d\xi_1$$

put $m = 1$

$$B_1(x,\xi) = C_1 K(x,\xi) - 1 \int_0^1 K(x,\xi_1) B_0(\xi_1,\xi)\, d\xi_1$$

$$B_1(x,\xi) = (1)(1) - \int_0^1 (1)(1)\, d\xi_1 = 0$$

$$C_2 = \int_0^1 B_1(\xi_1,\xi_1)\, d\xi_1 = \int_0^1 0 .\, d\xi_1 = 0 \Rightarrow \boxed{C_2 = 0}$$

$$B_2(x,\xi) = C_2 K(x,\xi) - 2\int_0^1 K(x,\xi_1) B_1(\xi_1,\xi)\, d\xi_1$$

$$= 0 - 2\int_0^1 1 . 0\, d\xi_1 = 0$$

So we have $C_1 = 1$, $C_2 = 0 = C_3 = - -$

$$B_0(x,\xi) = 1,\quad B_1(x,\xi) = 0 = B_2(x,\xi) = - -$$

$\therefore R(x,\xi;\lambda) = \dfrac{1-\lambda\cdot 0}{[1-\lambda\cdot(1)+0]} = \left(\dfrac{1}{1-\lambda}\right)$

$\therefore$ Required solⁿ of (1) is

$$\phi(x) = \sec^2 x + \lambda\int_0^1 \frac{1}{1-\lambda}\cdot \sec^2\xi \, d\xi \quad (\lambda \neq 1)$$

$$= \sec^2 x + \frac{\lambda}{1-\lambda}\int_0^1 \sec^2\xi \, d\xi$$

$$= \sec^2 x + \frac{\lambda}{1-\lambda}\left[\tan\xi\right]_0^1$$

$$\boxed{\phi(x) = \sec^2 x + \frac{\lambda}{1-\lambda}\cdot(\tan 1)}$$

(2) Solve the following

$$\phi(x) = e^x + \lambda\int_0^{10} x\xi\, \phi(\xi)\, d\xi$$

Solⁿ. Given

$$\phi(x) = e^x + \lambda\int_0^{10} x\xi\, \phi(\xi)\, d\xi \qquad \text{—— (1)}$$

Here $f(x) = e^x$, $K(x,\xi) = x\xi$

We know that the required solution of (1) is given by

$$\phi(x) = f(x) + \lambda\int_0^{10} R(x,\xi;\lambda) f(\xi)\, d\xi \qquad \text{— (2)}$$

where $f(x) = e^x \Rightarrow f(\xi) = e^2$

$R(x,\xi;\lambda) = \dfrac{D(x,\xi;\lambda)}{D(\lambda)}$ where

$$D(x,\xi;\lambda) = K(x,\xi) - \frac{\lambda}{\lfloor 1}B_1(x,\xi) + \frac{\lambda^2}{\lfloor 2}B_2(x,\xi) - \frac{\lambda^3}{\lfloor 3}B_3(x,\xi) + \cdots \qquad \text{— (3)}$$

$$D(\lambda) = 1 - \frac{\lambda}{\lfloor 1}c_1 + \frac{\lambda^2}{\lfloor 2}c_2 - \frac{\lambda^3}{\lfloor 3}c_3 + \cdots \qquad \text{— (4)}$$

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$$c_1 = \int_0^{10} B_0(\xi_1, \xi_1)\, d\xi_1 = \int_0^{10} \xi_1 . \xi_1\, d\xi_1 = \frac{1}{3}(10)^3$$

$, B_0(x,\xi) = K(x,\xi)$ (Here)

Now,

$$B_m(x,\xi) = c_m K(x,\xi) - m\int_0^{10} K(x,\xi_1)\, B_{m-1}(\xi_1,\xi)\, d\xi_1$$

put $m=1$

$$B_1(x;\xi) = c_1 K(x,\xi) - 1.\int_0^{10} K(x,\xi_1)\, B_0(\xi_1,\xi)\, d\xi_1$$

$$= \frac{10^3}{3} x\xi - \int_0^{10} x\xi_1 \cdot \xi_1 . \xi\, d\xi_1$$

$$= \frac{10^3}{3} x\xi - x\xi \int_0^{10} \xi_1^2\, d\xi_1$$

$$B_1(x,\xi) = \frac{10^3}{3} x\xi - \frac{x\xi}{3}\left[\xi_1^3\right]_0^{10}$$

$$= \left(\frac{10^3}{3} x\xi\right) - \left(\frac{10^3}{3} x\xi\right) = 0$$

$$c_2 = \int_0^{10} B_1(\xi_1,\xi_1)\, d\xi_1 = \int_0^{10} 0 . d\xi_1 = 0$$

$$\therefore B_2(x,\xi) = 0 = B_3(x,\xi) = \cdots$$

$$c_3 = 0 = c_4 = c_5 = \cdots,$$

$$\therefore D(x,\xi;\lambda) = K(x,\xi) - \lambda B_1(x,\xi) + \frac{\lambda^2}{\lfloor 2} B_2(x,\xi) - \frac{\lambda^3}{\lfloor 3} B_3(x,\xi) + \cdots$$

$$D(x,\xi;\lambda) = x\xi \quad [\because B_1(x,\xi) = 0 = B_2(x,\xi) = \cdots]$$

$$D(\lambda) = 1 - \lambda c_1 + \frac{\lambda^2}{\lfloor 2} c_2 - \frac{\lambda^3}{\lfloor 3} c_3 + \cdots$$

$$= \left(1 - \lambda . \frac{10^3}{3}\right) \quad [\because c_2 = 0 = c_3 = \cdots]$$

$$\therefore R(x,\xi;\lambda) = \frac{D(x,\xi;\lambda)}{D(\lambda)} = \left(\frac{x\xi}{1 - \frac{1000\lambda}{3}}\right)$$

$\therefore$ Required solution of ① is

$$\phi(x) = e^x + \lambda \int_0^{10} R(x,\xi;\lambda)\, e^{\xi}\, d\xi$$

$$= e^x + \lambda \int_0^{10} \frac{x\xi}{\left(1 - \frac{1000\lambda}{3}\right)} \cdot e^{\xi}\, d\xi$$

(4)

$$\phi(x) = e^x + \left(\frac{3\lambda x}{3-1000\lambda}\right)\int_0^{10} \xi \cdot e^{\xi}\, d\xi$$

$$\phi(x) = e^x + \frac{3\lambda x}{(3-1000\lambda)}\left[\xi \cdot e^{\xi}\Big|_0^{10} - \int_0^{10} 1 \cdot e^{\xi} d\xi\right]$$

$$\phi(x) = e^x + \frac{3\lambda x}{(3-1000\lambda)}\left[10e^{10} - (e^{\xi})_0^{10}\right]$$

$$\phi(x) = e^x + \frac{3\lambda x}{(3-1000\lambda)}\left[10e^{10} - (e^{10}-1)\right]$$

$$\phi(x) = e^x + \frac{3\lambda x}{(3-1000\lambda)}\left[9e^{10}+1\right]$$

(3) Solve the following :

$$\phi(x) = \sin x + \lambda\int_4^{10} x \cdot \phi(\xi)\, d\xi$$

Solution. Given

$$\phi(x) = \sin x + \lambda\int_4^{10} x\, \phi(\xi)\, d\xi \quad \text{--- (1)}$$

$\left[\text{Here } f(x) = \sin x,\ K(x,\xi) = x.\right]$

Its solution is given by

$$\phi(x) = f(x) + \lambda\int_4^{10} R(x,\xi;\lambda) f(\xi)\, d\xi \quad \text{--- (2)}$$

where $f(x) = \sin x \Rightarrow f(\xi) = \sin\xi$,

$B_0(x,\xi) = K_1(x,\xi) = K(x,\xi) = x$, $R(x,\xi;\lambda) = \frac{D(x,\xi;\lambda)}{D(\lambda)}$

Now,

$$D(x,\xi;\lambda) = K(x,\xi) - \frac{\lambda}{\lfloor 1} B_1(x,\xi) + \frac{\lambda^2}{\lfloor 2} B_2(x,\xi) - \frac{\lambda^3}{\lfloor 3} B_3(x,\xi) + \cdots \quad (3)$$

$$D(\lambda) = 1 - \frac{\lambda}{\lfloor 1} c_1 + \frac{\lambda^2}{\lfloor 2} c_2 - \frac{\lambda^3}{\lfloor 3} c_3 + \cdots \quad (4)$$

Now, $c_1 = \int_4^{10} B_0(\xi_1,\xi_1)\, d\xi_1 = \int_4^{10} \xi_1\, d\xi_1 = \frac{1}{2}\left[\xi_1^2\right]_4^{10} = \frac{1}{2}\left[10^2 - 4^2\right]$

$\boxed{c_1 = 42}$

We know that

$$B_m(x,\xi) = c_m K(x,\xi) - m\int_4^{10} K(x,\xi_1)\, B_{m-1}(\xi_1,\xi)\, d\xi_1 \quad \text{--- (5)}$$

put $m = 1$

$$B_1(x,\xi) = c_1 K(x,\xi) - 1.\int_4^{10} K(x,\xi_1)\, B_0(\xi_1,\xi)\, d\xi_1$$

$$B_1(x,\xi) = (42).x - \int_4^{10} x.\,\xi_1\, d\xi_1$$

$$\left[\because B_0(x,\xi) = K(x,\xi) = x\right]$$

$$= 42x - x.42 = 0$$

$$\Rightarrow B_1(x,\xi) = 0$$

$$c_2 = \int_4^{10} B_1(\xi_1,\xi_1)\, d\xi_1 = \int_4^{10} 0.\, d\xi_1 = 0$$

$$\Rightarrow B_2(x,\xi) = 0 = B_3(x,\xi) = 0 \cdots$$

and $c_3 = 0 = c_4 = \cdots$

$$\therefore D(x,\xi;\lambda) = K(x,\xi) - \lambda B_1(x,\xi) + \frac{\lambda^2}{\lfloor 2} B_2(x,\xi)$$

$$\Rightarrow D(x,\xi;\lambda) = x - 0 = x$$

$$D(\lambda) = 1 - \lambda c_1 + \frac{\lambda^2}{\lfloor 2} c_2 - \frac{\lambda^3}{\lfloor 3} c_3 + \cdots$$

$$\Rightarrow D(\lambda) = (1 - 42\lambda) \quad \left[\because c_2 = 0 = c_3 = \cdots\right]$$

$$\therefore R(x,\xi;\lambda) = \frac{D(x,\xi;\lambda)}{D(\lambda)} = \frac{x}{(1-42\lambda)}$$

$\therefore$ Required solution of (1) is

$$\phi(x) = \sin x + \lambda \int_4^{10} \frac{x}{(1-42\lambda)} . \sin\xi\, d\xi$$

$$= \sin x + \frac{\lambda x}{1-42\lambda}\left[-\cos\xi\right]_4^{10}$$

$$\phi(x) = \sin x - \frac{\lambda x}{(1-42\lambda)}\left[\cos 10 - \cos 4\right]$$

$$\phi(x) = \sin x - \left(\frac{2\lambda x}{1-42\lambda}\right)\left[\sin 7 \sin 3\right] \quad \left[\because \cos A - \cos B = 2\sin\frac{A+B}{2}\sin\frac{A-B}{2}\right]$$

(7) Solve the following

$$\phi(x) = e^x + \lambda \int_0^1 2 e^x . e^{\xi} \phi(\xi) d\xi$$

Solution. Given

$$\phi(x) = e^x + \lambda \int_0^1 2 e^x e^{\xi} \phi(\xi) d\xi$$

$$\phi(x) = e^x + 2\lambda \int_0^1 e^{x+\xi} \phi(\xi) d\xi \quad \text{--- (1)}$$

Here $f(x) = e^x$, $K(x,\xi) = 2e^{x+\xi}$

Its solution is given by

$$\phi(x) = e^x + \lambda \int_0^1 R(x,\xi;\lambda) e^{\xi} d\xi \quad \text{--- (2)}$$

where

$$R(x,\xi;\lambda) = \frac{D(x,\xi;\lambda)}{D(\lambda)}$$

$$D(x,\xi;\lambda) = K(x,\xi) - \frac{\lambda}{\underline{|1}} B_1(x,\xi) + \frac{\lambda^2}{\underline{|2}} B_2(x,\xi) - \frac{\lambda^3}{\underline{|3}} B_3(x,\xi) + \cdots \quad \text{--- (3)}$$

$$D(\lambda) = 1 - \lambda c_1 + \frac{\lambda^2}{\underline{|2}} c_2 - \frac{\lambda^3}{\underline{|3}} c_3 + \cdots \quad \text{--- (4)}$$

Now, $c_1 = \int_0^1 B_0(\xi_1,\xi_1) d\xi_1 = \int_0^1 K(\xi_1,\xi_1) d\xi_1 = 2.\int_0^1 e^{\xi_1+\xi_1} d\xi_1$

$$c_1 = 2 . \left[\frac{e^{2\xi_1}}{2}\right]_0^1 = \left[e^{2\xi_1}\right]_0^1 = (e^2-1)$$

We know that

$$B_m(x,\xi) = c_m K(x,\xi) - m \int_0^1 K(x,\xi_1) B_{m-1}(\xi_1,\xi) d\xi_1$$

put $m=1$

$$B_1(x,\xi) = c_1 K(x,\xi) - 1.\int_0^1 K(x,\xi_1) B_0(\xi_1,\xi) d\xi_1$$

$$= (e^2-1) 2 e^{x+\xi} - \int_0^1 2e^{x+\xi_1} . 2e^{\xi_1+\xi} d\xi_1$$

$$= (e^2-1)(2) e^{x+\xi} - 4 e^{x+\xi} \int_0^1 e^{2\xi_1} d\xi_1$$

$$= (e^2-1)(2) e^{x+\xi} - 2 e^{x+\xi} \left[e^{2\xi_1}\right]_0^1$$

$$B_1(x,\xi) = (e^2-1)(2) e^{x+\xi} - 2e^{x+\xi}[e^2-1] = 0$$

$\therefore$ $c_2 = 0 = c_3 = \cdots$

and $B_2(x,\xi) = 0 = B_3(x,\xi) = \cdots$

So,

$$D(x,\xi;\lambda) = K(x,\xi) - \frac{\lambda}{\lfloor 1} B_1(x,\xi) + \frac{\lambda^2}{\lfloor 2} B_2(x,\xi) - \cdots$$

$$D(x,\xi;\lambda) = 2e^{x+\xi} \quad [\because B_1(x,\xi) = 0 = B_2(x,\xi) = \cdots]$$

$$D(\lambda) = 1 - \lambda c_1 + \frac{\lambda^2}{\lfloor 2} c_2 - \cdots$$

$$D(\lambda) = 1 - \lambda\,(e^2-1) + 0$$

$$\therefore R(x,\xi;\lambda) = \frac{D(x,\xi;\lambda)}{D(\lambda)} = \frac{2e^{x+\xi}}{1-\lambda(e^2-1)}$$

$\therefore$ Required solution of (1) is given by

$$\phi(x) = e^x + \lambda \int_0^1 \frac{2e^{x+\xi}}{1-\lambda(e^2-1)} \cdot e^{\xi}\, d\xi$$

$$\phi(x) = e^x + \frac{2\lambda e^x}{1-\lambda(e^2-1)} \int_0^1 e^{2\xi}\, d\xi$$

$$\phi(x) = e^x + \frac{\lambda e^x}{1-\lambda(e^2-1)} \left[e^{2\xi} \right]_0^1$$

$$\phi(x) = e^x + \frac{\lambda e^x (e^2-1)}{1-\lambda(e^2-1)}$$