SUBJECT: MATHEMATICS

PAPER: ADVANCED MATHEMATICAL METHODS
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(5) Find Fredholm determinant and Resolvent Kernel of $K(x,\xi) = \sin x - \sin\xi$, $a=0$, $b=2\pi$.

Solution. We know Fredholm determinant

$$D(\lambda) = 1 - \frac{\lambda}{\lfloor 1} c_1 + \frac{\lambda^2}{\lfloor 2} c_2 - \frac{\lambda^3}{\lfloor 3} c_3 + \cdots \quad (1)$$

and Resolvent Kernel

$$R(x,\xi;\lambda) = \frac{D(x,\xi;\lambda)}{D(\lambda)} \quad (2)$$

where $D(x,\xi;\lambda) = B_0(x,\xi) - \frac{\lambda}{\lfloor 1} B_1(x,\xi) + \frac{\lambda^2}{\lfloor 2} B_2(x,\xi) - \frac{\lambda^3}{\lfloor 3} B_3(x,\xi) + \cdots \quad (3)$

Here $B_0(x,\xi) = K(x,\xi) = \sin x - \sin\xi$ (given). $\quad (4)$

Now,

$$B_1(x,\xi) = \int_0^{2\pi} K\begin{pmatrix} x & \xi_1 \\ \xi & \xi_1 \end{pmatrix} d\xi_1 = \int_0^{2\pi} \begin{vmatrix} K(x,\xi) & K(x,\xi_1) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) \end{vmatrix} d\xi_1 \quad (5)$$

let $D_1 = \begin{vmatrix} K(x,\xi) & K(x,\xi_1) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) \end{vmatrix}$

$$= \begin{vmatrix} \sin x - \sin\xi & \sin x - \sin\xi_1 \\ \sin\xi_1 - \sin\xi & \sin\xi_1 - \sin\xi_1 \end{vmatrix}$$

$$= \begin{vmatrix} \sin x - \sin\xi & \sin x - \sin\xi_1 \\ \sin\xi_1 - \sin\xi & 0 \end{vmatrix}$$

$$= -(\sin\xi_1 - \sin\xi)(\sin x - \sin\xi_1)$$

$$= -[\sin x \sin\xi_1 - \sin^2\xi_1 - \sin x \sin\xi + \sin\xi \sin\xi_1]$$

$$\therefore B_1(x,\xi) = -\sin x \int_0^{2\pi} \sin\xi_1 \, d\xi_1 + \int_0^{2\pi} \sin^2\xi_1 \, d\xi_1 + \sin x \sin\xi \int_0^{2\pi} d\xi_1 + \sin\xi \cdot \int_0^{2\pi} \sin\xi_1 \, d\xi_1$$

$$= 0 + \frac{1}{2}\int_0^{2\pi} (1 - \cos 2\xi_1)\, d\xi_1 + \sin x \sin\xi \cdot 2\pi + 0$$

$$B_1(x,\xi) = \pi + 2\pi \sin x \sin\xi = \pi[1 + 2\sin x \sin\xi]$$

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$$B_2(x,\xi) = \int_0^{2\pi}\int_0^{2\pi} K\begin{pmatrix} x & \xi_1 & \xi_2 \\ \xi & \xi_1 & \xi_2 \end{pmatrix} d\xi_1\, d\xi_2 = \int_0^{2\pi}\int_0^{2\pi} D_2(x,\xi_1,\xi_2)\, d\xi_1\, d\xi_2$$

where $D_2(x,\xi_1,\xi_2) = K\begin{pmatrix} x & \xi_1 & \xi_2 \\ \xi & \xi_1 & \xi_2 \end{pmatrix}$

$$= \begin{vmatrix} K(x,\xi) & K(x,\xi_1) & K(x,\xi_2) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) & K(\xi_1,\xi_2) \\ K(\xi_2,\xi) & K(\xi_2,\xi_1) & K(\xi_2,\xi_2) \end{vmatrix}$$

$$= \begin{vmatrix} \sin x - \sin\xi & \sin x - \sin\xi_1 & \sin x - \sin\xi_2 \\ \sin\xi_1 - \sin\xi & \sin\xi_1 - \sin\xi_1 & \sin\xi_1 - \sin\xi_2 \\ \sin\xi_2 - \sin\xi & \sin\xi_2 - \sin\xi_1 & \sin\xi_2 - \sin\xi_2 \end{vmatrix}$$

$$= \begin{vmatrix} \sin x - \sin\xi & \sin\xi - \sin\xi_1 & \sin\xi - \sin\xi_2 \\ \sin\xi_1 - \sin\xi & \sin\xi - \sin\xi_1 & \sin\xi - \sin\xi_2 \\ \sin\xi_2 - \sin\xi & \sin\xi - \sin\xi_1 & \sin\xi - \sin\xi_2 \end{vmatrix} \quad \begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{matrix}$$

$$= (\sin\xi - \sin\xi_1)(\sin\xi - \sin\xi_2) \begin{vmatrix} \sin x - \sin\xi & 1 & 1 \\ \sin\xi_1 - \sin\xi & 1 & 1 \\ \sin\xi_1 - \sin\xi & 1 & 1 \end{vmatrix}$$

$= (\sin\xi - \sin\xi_1)(\sin\xi - \sin\xi_2) \times 0$, ($C_2$ and C_3 are equal.)

$D_2(x,\xi_1,\xi_2) = 0$

$\therefore B_2(x,\xi) = 0$

Similarly, $B_3(x,\xi) = 0 = B_4(x,\xi) = \ldots$

Now,
$$c_1 = \int_0^{2\pi} K(\xi_1, \xi_1)\, d\xi_1 = \int_0^{2\pi} (\sin\xi_1 - \sin\xi_1)\, d\xi_1 = 0$$

$$c_2 = \int_0^{2\pi}\int_0^{2\pi} K\begin{pmatrix} \xi_1 & \xi_2 \\ \xi_1 & \xi_2 \end{pmatrix} d\xi_1 d\xi_2 = \int_0^{2\pi}\int_0^{2\pi} \begin{vmatrix} K(\xi_1, \xi_1) & K(\xi_1, \xi_2) \\ K(\xi_2, \xi_1) & K(\xi_2, \xi_2) \end{vmatrix} d\xi_1 d\xi_2$$

$$c_2 = \int_0^{2\pi}\int_0^{2\pi} \begin{vmatrix} \sin\xi_1 - \sin\xi_1 & \sin\xi_1 - \sin\xi_2 \\ \sin\xi_2 - \sin\xi_1 & \sin\xi_2 - \sin\xi_2 \end{vmatrix} d\xi_1 . d\xi_2$$

$$c_2 = \int_0^{2\pi}\int_0^{2\pi} \begin{vmatrix} 0 & \sin\xi_1 - \sin\xi_2 \\ -(\sin\xi_1 - \sin\xi_2) & 0 \end{vmatrix} d\xi_1 d\xi_2$$

$$c_2 = \int_0^{2\pi}\int_0^{2\pi} (\sin\xi_1 - \sin\xi_2)^2 d\xi_1 d\xi_2$$

$$c_2 = \int_0^{2\pi}\int_0^{2\pi} (\sin^2\xi_1 + \sin^2\xi_2 - 2\sin\xi_1 \sin\xi_2) d\xi_1 d\xi_2$$

$$c_2 = \int_0^{2\pi}\int_0^{2\pi} \left(\frac{1-\cos 2\xi_1}{2} + \frac{1-\cos 2\xi_2}{2} - 2\sin\xi_1 \sin\xi_2\right) d\xi_1 d\xi_2$$

$$c_2 = \int_0^{2\pi}\int_0^{2\pi} \left[1 - 2\sin\xi_1 \sin\xi_2 - \frac{1}{2}(\cos 2\xi_1 + \cos 2\xi_2)\right] d\xi_1 d\xi_2$$

$$c_2 = \int_0^{2\pi}\int_0^{2\pi} d\xi_1 d\xi_2 - 2\int_0^{2\pi}\int_0^{2\pi} \sin\xi_1 \sin\xi_2\, d\xi_1 d\xi_2 - \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} (\cos 2\xi_1 + \cos 2\xi_2) \times d\xi_1 d\xi_2$$

$$c_2 = \int_0^{2\pi} 2\pi . d\xi_2 - 0 - 0$$

$$\boxed{c_2 = 4\pi^2}$$

$$c_3 = \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} K\begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \\ \xi_1 & \xi_2 & \xi_3 \end{pmatrix} d\xi_1\, d\xi_2\, d\xi_3$$

$$= \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} D_3(\xi_1, \xi_2, \xi_3)\, d\xi_1\, d\xi_2\, d\xi_3$$

where

$$D_3(\xi_1, \xi_2, \xi_3) = K\begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \\ \xi_1 & \xi_2 & \xi_3 \end{pmatrix} = \begin{vmatrix} K(\xi_1, \xi_1) & K(\xi_1, \xi_2) & K(\xi_1, \xi_3) \\ K(\xi_2, \xi_1) & K(\xi_2, \xi_2) & K(\xi_2, \xi_3) \\ K(\xi_3, \xi_1) & K(\xi_3, \xi_2) & K(\xi_3, \xi_3) \end{vmatrix}$$

$$D_3 = \begin{vmatrix} \sin\xi_1 - \sin\xi_1 & \sin\xi_1 - \sin\xi_2 & \sin\xi_1 - \sin\xi_3 \\ \sin\xi_2 - \sin\xi_1 & \sin\xi_2 - \sin\xi_2 & \sin\xi_2 - \sin\xi_3 \\ \sin\xi_3 - \sin\xi_1 & \sin\xi_3 - \sin\xi_2 & \sin\xi_3 - \sin\xi_3 \end{vmatrix}$$

$$= \begin{vmatrix} \sin\xi_1 - \sin\xi_1 & \sin\xi_1 - \sin\xi_2 & \sin\xi_1 - \sin\xi_3 \\ \sin\xi_2 - \sin\xi_1 & \sin\xi_1 - \sin\xi_2 & \sin\xi_1 - \sin\xi_3 \\ \sin\xi_3 - \sin\xi_1 & \sin\xi_1 - \sin\xi_2 & \sin\xi_1 - \sin\xi_3 \end{vmatrix} \quad \begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{matrix}$$

$$= (\sin\xi_1 - \sin\xi_2)(\sin\xi_1 - \sin\xi_3) \begin{vmatrix} \sin\xi_1 - \sin\xi_1 & 1 & 1 \\ \sin\xi_2 - \sin\xi_1 & 1 & 1 \\ \sin\xi_3 - \sin\xi_1 & 1 & 1 \end{vmatrix}$$

$$= (\sin\xi_1 - \sin\xi_2)(\sin\xi_1 - \sin\xi_3) \times 0 \quad [\because c_1 \,\&\, c_2 \text{ are equal}]$$

$$D_3 = 0$$

$$\therefore\; c_3 = \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} 0 \cdot d\xi_1\, d\xi_2\, d\xi_3 = 0$$

Similarly $c_4 = 0 = c_5 = \cdots$

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$\because D(x,\xi;\lambda) = B_0(x,\xi) - \frac{\lambda}{\lfloor 1} B_1(x,\xi) + \frac{\lambda^2}{\lfloor 2} B_2(x,\xi)$

$- \frac{\lambda^3}{\lfloor 3} B_3(x,\xi) + \frac{\lambda^4}{\lfloor 4} B_4(x,\xi) - \cdots$

$D(x,\xi;\lambda) = \sin x - \sin \xi - \lambda . \pi (1 + 2 \sin x \sin \xi) + 0 - 0 + 0$

$[\because B_0(x,\xi) = K(x,\xi)]$

$D(\lambda) = 1 - \frac{\lambda}{\lfloor 1} c_1 + \frac{\lambda^2}{\lfloor 2} c_2 - \frac{\lambda^3}{\lfloor 3} c_3 + \frac{\lambda^4}{\lfloor 4} c_4 - \cdots$

$= 1 - \lambda . 0 + \frac{\lambda^2}{\lfloor 2} . (4\pi^2) - 0 + 0 \cdots$

$D(\lambda) = 1 + \frac{\lambda^2}{2}(4\pi^2) = 1 + 2\pi^2\lambda^2$

$\because R(x,\xi;\lambda) = \frac{D(x,\xi;\lambda)}{D(\lambda)}$

$R(x,\xi;\lambda) = \left[\frac{\sin x - \sin \xi - \lambda\pi (1 + 2 \sin x \sin \xi)}{1 + 2\pi^2\lambda^2} \right]$

which is required resolvent kernel.

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(6) Find $R(x, \xi; \lambda)$ of the following

$$K(x, \xi) = 1 + 3x\xi, \quad 0 \le x \le 1, \quad 0 \le \xi \le 1$$

Solution. Given

$$B_0(x, \xi) = K(x, \xi) = 1 + 3x\xi$$

$$B_1(x, \xi) = \int_0^1 K\begin{pmatrix} x & \xi_1 \\ \xi & \xi_1 \end{pmatrix} d\xi_1 = \int_0^1 \begin{vmatrix} K(x, \xi) & K(x, \xi_1) \\ K(\xi_1, \xi) & K(\xi_1, \xi_1) \end{vmatrix} d\xi_1$$

$$B_1(x, \xi) = \int_0^1 \begin{vmatrix} 1 + 3x\xi & 1 + 3x\xi_1 \\ 1 + 3\xi_1\xi & 1 + 3\xi_1\xi_1 \end{vmatrix} d\xi_1$$

$$B_1(x, \xi) = \int_0^1 \begin{vmatrix} 1 + 3x\xi & 3x\xi_1 - 3x\xi \\ 1 + 3\xi_1\xi & 3\xi_1^2 - 3\xi_1\xi \end{vmatrix} d\xi_1 \qquad C_2 \to C_2 - C_1$$

$$B_1(x, \xi) = \int_0^1 \begin{vmatrix} 1 + 3x\xi & 3x(\xi_1 - \xi) \\ 1 + 3\xi_1\xi & 3\xi_1(\xi_1 - \xi) \end{vmatrix} d\xi_1$$

$$B_1(x, \xi) = \int_0^1 3(\xi_1 - \xi) \begin{vmatrix} 1 + 3x\xi & x \\ 1 + 3\xi_1\xi & \xi_1 \end{vmatrix} d\xi_1.$$

$$B_1(x, \xi) = \int_0^1 3(\xi_1 - \xi)(\xi_1 + \cancel{3x\xi\xi_1} - x - \cancel{3\xi_1\xi x})\, d\xi_1$$

$$B_1(x, \xi) = \int_0^1 3(\xi_1 - \xi)(\xi_1 - x)\, d\xi_1$$

$$B_1(x, \xi) = 3\int_0^1 \xi_1^2 - x\xi_1 - \xi\xi_1 + x\xi \quad d\xi_1$$

$$B_1(x, \xi) = 3\left[\frac{1}{3} - \frac{x}{2} - \frac{\xi}{2} + x\xi\right] = 3\left[x\xi - \left(\frac{x+\xi}{2}\right) + \frac{1}{3}\right]$$

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$$B_2(x,\xi) = \int_0^1 \int_0^1 K\begin{pmatrix} x & \xi_1 & \xi_2 \\ \xi & \xi_1 & \xi_2 \end{pmatrix} d\xi_1 \, d\xi_2$$

$$= \int_0^1 \int_0^1 D_2 \, d\xi_1 \, d\xi_2$$

where

$$D_2 = K\begin{pmatrix} x & \xi_1 & \xi_2 \\ \xi & \xi_1 & \xi_2 \end{pmatrix} = \begin{vmatrix} K(x,\xi) & K(x,\xi_1) & K(x,\xi_2) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) & K(\xi_1,\xi_2) \\ K(\xi_2,\xi) & K(\xi_2,\xi_1) & K(\xi_2,\xi_2) \end{vmatrix}$$

$$= \begin{vmatrix} 1+3x\xi & 1+3x\xi_1 & 1+3x\xi_2 \\ 1+3\xi_1\xi & 1+3\xi_1\xi_1 & 1+3\xi_1\xi_2 \\ 1+3\xi_2\xi & 1+3\xi_2\xi_1 & 1+3\xi_2\xi_2 \end{vmatrix}$$

$$= \begin{vmatrix} 1+3x\xi & 3x(\xi_1-\xi) & 3x(\xi_2-\xi) \\ 1+3\xi_1\xi & 3\xi_1(\xi_1-\xi) & 3\xi_1(\xi_2-\xi) \\ 1+3\xi_2\xi & 3\xi_2(\xi_1-\xi) & 3\xi_2(\xi_2-\xi) \end{vmatrix} \quad \begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{matrix}$$

$$= (\xi_1-\xi)(\xi_2-\xi) \begin{vmatrix} 1+3x\xi & 3x & 3x \\ 1+3\xi_1\xi & 3\xi_1 & 3\xi_1 \\ 1+3\xi_2\xi & 3\xi_2 & 3\xi_2 \end{vmatrix}$$

$D_2 = 0$ as C_2 and C_3 are equal

$$\therefore B_2(x,\xi) = \int_0^1 \int_0^1 0 \cdot d\xi_1 \, d\xi_2 = 0$$

Similarly $B_3(x,\xi) = 0 = B_4(x,\xi) = \cdots$

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Also $\quad c_1 = \int_0^1 K(\xi_1, \xi_1)\, d\xi_1 = \int_0^1 (1 + 3\xi_1\xi_1)\, d\xi_1$

$$c_1 = \int_0^1 d\xi_1 + 3\int_0^1 \xi_1^2\, d\xi_1 = 1 + 1 = 2$$

$$\Rightarrow \boxed{c_1 = 2}$$

$$c_2 = \int_0^1\int_0^1 K\begin{pmatrix} \xi_1 & \xi_2 \\ \xi_1 & \xi_2 \end{pmatrix} d\xi_1\, d\xi_2 = \int_0^1\int_0^1 \begin{vmatrix} K(\xi_1, \xi_1) & K(\xi_1, \xi_2) \\ K(\xi_2, \xi_1) & K(\xi_2, \xi_2) \end{vmatrix} d\xi_1\, d\xi_2$$

$$= \int_0^1\int_0^1 \begin{vmatrix} 1+3\xi_1^2 & 1+3\xi_1\xi_2 \\ 1+3\xi_2\xi_1 & 1+3\xi_2^2 \end{vmatrix} d\xi_1\, d\xi_2$$

$$= \int_0^1\int_0^1 \left[(1+3\xi_1^2)(1+3\xi_2^2) - (1+3\xi_1\xi_2)^2\right] d\xi_1\, d\xi_2$$

$$= \int_0^1\int_0^1 (1 + 3\xi_1^2 + 3\xi_2^2 + 9\xi_1^2\xi_2^2 - 1 - 9\xi_1^2\xi_2^2 - 6\xi_1\xi_2)\, d\xi_1\, d\xi_2$$

$$= \int_0^1 (1 + \cancel{1} + \cancel{3\xi_2^2} + 3\xi_2^2 - \cancel{1} - \cancel{3\xi_2^2} - 3\xi_2)\, d\xi$$

$$c_2 = \int_0^1 (1 + 3\xi_2^2 - 3\xi_2)\, d\xi_2 = 1 + 1 - \frac{3}{2} = \frac{1}{2}$$

$$\Rightarrow c_2 = \frac{1}{2}$$

$$c_3 = \int_0^1\int_0^1\int_0^1 K\begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \\ \xi_1 & \xi_2 & \xi_3 \end{pmatrix} d\xi_1\, d\xi_2\, d\xi_3 = \int_0^1\int_0^1\int_0^1 D_3\, d\xi_1\, d\xi_2\, d\xi_3$$

$$, \quad D_3 = K\begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \\ \xi_1 & \xi_2 & \xi_3 \end{pmatrix}$$

where $D_3 = \begin{vmatrix} K(\xi_1, \xi_1) & K(\xi_1, \xi_2) & K(\xi_1, \xi_3) \\ K(\xi_2, \xi_1) & K(\xi_2, \xi_2) & K(\xi_2, \xi_3) \\ K(\xi_3, \xi_1) & K(\xi_3, \xi_2) & K(\xi_3, \xi_3) \end{vmatrix}$

$$D_3 = \begin{vmatrix} 1+3\xi_1^2 & 1+3\xi_1\xi_2 & 1+3\xi_1\xi_3 \\ 1+3\xi_1\xi_2 & 1+3\xi_2^2 & 1+3\xi_2\xi_3 \\ 1+3\xi_1\xi_3 & 1+3\xi_2\xi_3 & 1+3\xi_3^2 \end{vmatrix}$$

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$$D_3 = \begin{vmatrix} 1+3\xi_1^2 & 3\xi_1(\xi_2-\xi_1) & 3\xi_1(\xi_3-\xi_1) \\ 1+3\xi_1\xi_2 & 3\xi_2(\xi_2-\xi_1) & 3\xi_2(\xi_3-\xi_1) \\ 1+3\xi_1\xi_3 & 3\xi_3(\xi_2-\xi_1) & 3\xi_3(\xi_3-\xi_1) \end{vmatrix} \begin{matrix} c_2 \to c_2 - c_1 \\ c_3 \to c_3 - c_1 \end{matrix}$$

$$= 9(\xi_2-\xi_1)(\xi_3-\xi_1) \begin{vmatrix} 1+3\xi_1^2 & \xi_1 & \xi_1 \\ 1+3\xi_1\xi_2 & \xi_2 & \xi_2 \\ 1+3\xi_1\xi_3 & \xi_3 & \xi_3 \end{vmatrix}$$

$D_3 = 9(\xi_2-\xi_1)(\xi_3-\xi_1) \times 0$ as two columns are equal

$D_3 = 0$

$\therefore\ C_3 = \int_0^1\int_0^1\int_0^1 D_3\, d\xi_1\, d\xi_2\, d\xi_3 = 0$ as $D_3 = 0.$

Similarly $C_4 = 0 = C_5 = \cdots\cdots$

$$\therefore\ D(x,\xi;\lambda) = K(x,\xi) - \frac{\lambda}{\underline{1}} B_1(x,\xi) + \frac{\lambda^2}{\underline{2}} B_2(x,\xi) - \frac{\lambda^3}{\underline{3}} B_3(x,\xi) + \cdots$$

$$D(x,\xi;\lambda) = 1+3x\xi - 3\lambda\left\{x\xi - \left(\frac{x+\xi}{2}\right) + \frac{1}{3}\right\} + 0.$$

$$D(\lambda) = 1 - \frac{\lambda}{\underline{1}} C_1 + \frac{\lambda^2}{\underline{2}} C_2 - \frac{\lambda^3}{\underline{3}} C_3 + \cdots$$

$$= 1 - \frac{\lambda}{\underline{1}}\cdot 2 + \frac{\lambda^2}{2}\cdot\frac{1}{2} - 0$$

$$D(\lambda) = \left(1 - 2\lambda + \frac{\lambda^2}{4}\right)$$

$$\therefore\ R(x,\xi;\lambda) = \frac{D(x,\xi;\lambda)}{D(\lambda)} = \frac{1+3x\xi - 3\lambda\left\{x\xi - \left(\frac{x+\xi}{2}\right) + \frac{1}{3}\right\}}{\left(1-2\lambda+\frac{\lambda^2}{4}\right)}.$$

Q 07: Find the resolvent Kernel of the following kernel $K(x,\xi) = x - 2\xi$, $0 \le x \le 1, 0 \le \xi \le 1$

Solution. Given $K(x,\xi) = x - 2\xi$, $a = 0, b = 1$

To find $R(x,\xi;\lambda)$.

We know that $B_0(x,\xi) = K(x,\xi) = x - 2\xi$

$$B_1(x,\xi) = \int_0^1 K\begin{pmatrix} x & \xi_1 \\ \xi & \xi_1 \end{pmatrix} d\xi_1 = \int_0^1 \begin{vmatrix} K(x,\xi) & K(x,\xi_1) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) \end{vmatrix} d\xi_1$$

$$= \int_0^1 \begin{vmatrix} x-2\xi & x-2\xi_1 \\ \xi_1 - 2\xi & \xi_1 - 2\xi_1 \end{vmatrix} d\xi_1 = \int_0^1 \begin{vmatrix} x-2\xi & 2(\xi-\xi_1) \\ \xi_1 - 2\xi & 2(\xi-\xi_1) \end{vmatrix} d\xi_1 , \; C_2 \to C_2 - C_1$$

$$= 2\int_0^1 \begin{vmatrix} x-2\xi & \xi-\xi_1 \\ \xi_1-2\xi & \xi-\xi_1 \end{vmatrix} d\xi_1$$

$$= 2\int_0^1 \left[(x-2\xi)(\xi-\xi_1) - (\xi_1-2\xi)(\xi-\xi_1)\right] d\xi_1$$

$$= 2\int_0^1 (\xi-\xi_1)\left[x - \cancel{2\xi} - \xi_1 + \cancel{2\xi}\right] d\xi_1$$

$$B_1(x,\xi) = 2\int_0^1 (\xi-\xi_1)(x-\xi_1)\, d\xi_1 = 2\int_0^1 (x\xi - x\xi_1 - \xi\xi_1 + \xi_1^2)\, d\xi_1$$

$$B_1(x,\xi) = 2\left[x\xi - \frac{x}{2} - \frac{\xi}{2} + \frac{1}{3}\right] = 2\left[x\xi - \left(\frac{x+\xi}{2}\right) + \frac{1}{3}\right]$$

$$B_2(x,\xi) = \int_0^1\int_0^1 K\begin{pmatrix} x & \xi_1 & \xi_2 \\ \xi & \xi_1 & \xi_2 \end{pmatrix} d\xi_1\, d\xi_2 = \int_0^1\int_0^1 \begin{vmatrix} K(x,\xi) & K(x,\xi_1) & K(x,\xi_2) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) & K(\xi_1,\xi_2) \\ K(\xi_2,\xi) & K(\xi_2,\xi_1) & K(\xi_2,\xi_2) \end{vmatrix} \times d\xi_1\, d\xi_2$$

$$= \int_0^1\int_0^1 D_2 \cdot d\xi_1\, d\xi_2$$

where

$$D = \begin{vmatrix} K(x,\xi) & K(x,\xi_1) & K(x,\xi_2) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) & K(\xi_1,\xi_2) \\ K(\xi_2,\xi) & K(\xi_2,\xi_1) & K(\xi_2,\xi_2) \end{vmatrix}$$

$$D = \begin{vmatrix} x-2\xi & x-2\xi_1 & x-2\xi_2 \\ \xi_1-2\xi & \xi_1-2\xi_1 & \xi_1-2\xi_2 \\ \xi_2-2\xi & \xi_2-2\xi_1 & \xi_2-2\xi_2 \end{vmatrix}$$

$$= \begin{vmatrix} x-2\xi & 2(\xi-\xi_1) & 2(\xi-\xi_2) \\ \xi_1-2\xi & 2(\xi-\xi_1) & 2(\xi-\xi_2) \\ \xi_2-2\xi & 2(\xi-\xi_1) & 2(\xi-\xi_2) \end{vmatrix}$$

$$= (\xi-\xi_1)(\xi-\xi_2) \begin{vmatrix} x-2\xi & 2 & 2 \\ \xi_1-2\xi & 2 & 2 \\ \xi_2-2\xi & 2 & 2 \end{vmatrix}$$

$= (\xi-\xi_1)(\xi-\xi_2) \times 0$ as two columns are equal.

$D = 0$

$\therefore\ B_2(x,\xi) = \int_0^1 \int_0^1 0 \cdot d\xi_1 d\xi_2 = 0.$

Similarly $B_3(x,\xi) = 0 = B_4(x,\xi) = \ldots\ldots$

$c_1 = \int_0^1 K(\xi_1,\xi_1)\, d\xi_1 = \int_0^1 (\xi_1 - 2\xi_1)\, d\xi_1$

$c_1 = \int_0^1 (-\xi_1)\, d\xi_1 \Rightarrow \boxed{c_1 = -\frac{1}{2}}$

[c_1 can also be find out by the formula

$c_m = \int_0^1 B_{m-1}(\xi_1,\xi_1)\, d\xi_1$

Put $m=1$

$c_1 = \int_0^1 B_0(\xi_1,\xi_1)\, d\xi_1 = \int_0^1 K(\xi_1,\xi_1)\, d\xi_1$

$c_1 = \int_0^1 (\xi_1 - 2\xi_1)\, d\xi_1 = \int_0^1 (-\xi_1)\, d\xi_1 = -\frac{1}{2}$]

$$c_2 = \int_0^1 B_1(\xi_1, \xi_1)\, d\xi_1 \quad , B_1(x,\xi) = 2\left[x\xi - \left(\frac{x+\xi}{2}\right) + \frac{1}{3}\right]$$

$$c_2 = \int_0^1 2\left\{ \xi_1 \xi_1 - \frac{\xi_1 + \xi_1}{2} + \frac{1}{3} \right\} d\xi_1$$

$$c_2 = \int_0^1 \left(2\xi_1^2 - 2\xi_1 + \frac{2}{3}\right) d\xi_1$$

$$c_2 = \frac{2}{3} - 1 + \frac{2}{3} = \frac{4}{3} - 1 = \frac{1}{3}$$

$$c_3 = \int_0^1 B_2(\xi_1, \xi_1)\, d\xi_1 \qquad B_2(x,\xi) = 0$$

$$\Rightarrow B_2(\xi_1, \xi_1) = 0$$

$$\Rightarrow c_3 = \int_0^1 0 .\, d\xi_1 = 0$$

Similarly $c_4 = 0 = c_5 = \cdots\cdots$

$$\therefore D(x,\xi;\lambda) = K(x,\xi) - \frac{\lambda}{\lfloor 1} B_1(x,\xi) + \frac{\lambda^2}{\lfloor 2} B_2(x,\xi) + \cdots$$

$$= x - 2\xi - 2\lambda\left(x\xi - \left(\frac{x+\xi}{2}\right) + \frac{1}{3}\right) + 0.$$

$$D(\lambda) = 1 - \frac{\lambda}{\lfloor 1} c_1 + \frac{\lambda^2}{\lfloor 2} c_2 - \frac{\lambda^3}{\lfloor 3} c_3 + \cdots$$

$$= 1 - \lambda . \left(-\frac{1}{2}\right) + \frac{\lambda^2}{\lfloor 2}\left(\frac{1}{3}\right) - 0.$$

$$D(\lambda) = \left(1 + \frac{\lambda}{2} + \frac{\lambda^2}{6}\right)$$

$$\therefore R(x,\xi;\lambda) = \frac{D(x,\xi;\lambda)}{D(\lambda)}$$

$$\boxed{R(x,\xi;\lambda) = \frac{\left[x - 2\xi - 2\lambda\left[x\xi - \left(\frac{x+\xi}{2}\right) + \frac{1}{3}\right]\right]}{\left[1 + \frac{\lambda}{2} + \frac{\lambda^2}{6}\right]}}$$

(8) Find the resolvent kernel of

$$K(x,\xi) = x+\xi+1, \quad -1 \le x \le 1, \quad -1 \le \xi \le 1$$

Solution. Given $K(x,\xi) = x+\xi+1$, $a=-1$, $b=1$

To find $R(x,\xi;\lambda)$.

We know that $B_0(x,\xi) = K(x,\xi) = x+\xi+1$

Now,

$$B_1(x,\xi) = \int_{-1}^{1} K\begin{pmatrix} x & \xi_1 \\ \xi & \xi_1 \end{pmatrix} d\xi_1 = \int_{-1}^{1} \begin{vmatrix} K(x,\xi) & K(x,\xi_1) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) \end{vmatrix} d\xi_1$$

$$= \int_{-1}^{1} \begin{vmatrix} x+\xi+1 & x+\xi_1+1 \\ \xi_1+\xi+1 & \xi_1+\xi_1+1 \end{vmatrix} d\xi_1$$

$$= \int_{-1}^{1} \begin{vmatrix} x+\xi+1 & \xi_1-\xi \\ \xi_1+\xi+1 & \xi_1-\xi \end{vmatrix} d\xi_1, \quad C_2 \to C_2 - C_1$$

$$= \int_{-1}^{1} (\xi_1-\xi) \begin{vmatrix} x+\xi+1 & 1 \\ \xi_1+\xi+1 & 1 \end{vmatrix} d\xi_1$$

$$= \int_{-1}^{1} (\xi_1-\xi)(x+\cancel{\xi}+\underline{1}-\xi_1-\cancel{\xi}-\underline{1})\, d\xi_1$$

$$= \int_{-1}^{1} (\xi_1-\xi)(x-\xi_1)\, d\xi_1$$

$$= \int_{-1}^{1} (x\xi_1 - x\xi - \xi_1^2 + \xi\xi_1)\, d\xi_1$$

$$= x\left[\frac{\xi_1^2}{2}\right]_{-1}^{1} - x\xi\left[\xi_1\right]_{-1}^{1} - \frac{1}{3}\left[\xi_1^3\right]_{-1}^{1} + \frac{\xi}{2}\left[\xi_1^2\right]_{-1}^{1}$$

$$B_1(x,\xi) = 0 - 2x\xi - \frac{2}{3} + 0 = -2\left[x\xi + \frac{1}{3}\right]$$

$$C_1 = \int_{-1}^{1} B_0(\xi_1,\xi_1)\, d\xi_1 = \int_{-1}^{1} (\xi_1+\xi_1+1)\, d\xi_1$$

$$C_1 = \int_{-1}^{1} (2\xi_1+1)\, d\xi_1 = \left[\xi_1^2\right]_{-1}^{1} + \left[\xi_1\right]_{-1}^{1} = 0+2$$

$$C_1 = 2$$

Relation between C_m and $B_m(x, \xi)$

$$B_m(x,\xi) = C_m K(x,\xi) - m\int_a^b K(x,\xi_1) B_{m-1}(\xi_1,\xi)\, d\xi_1$$

For $m=2$

$$B_2(x,\xi) = C_2 K(x,\xi) - 2\int_{-1}^{1} K(x,\xi_1) B_1(\xi_1,\xi)\, d\xi_1$$

Now first we find $\quad \left[B_1(x,\xi) = -2\left[x\xi + \frac{1}{3}\right]\right.$

$$C_2 = \int_{-1}^{1} B_1(\xi_1,\xi_1)\, d\xi_1$$

$$= -2\int_{-1}^{1} \left(\xi_1^2 + \frac{1}{3}\right) d\xi_1 = -2\left[\frac{1}{3}(\xi_1^3)_{-1}^{1} + \frac{1}{3}(\xi_1)_{-1}^{1}\right]$$

$$C_2 = -2\left[\frac{2}{3} + \frac{2}{3}\right] = -\frac{8}{3}$$

$$\therefore B_2(x,\xi) = \left(-\frac{8}{3}\right)\left[x+\xi+1\right] - 2\int_{-1}^{1} (x+\xi_1+1)\left[-2\left(\xi_1\xi + \frac{1}{3}\right)\right] d\xi_1$$

$$= -\frac{8}{3}(x+\xi+1) + 4\int_{-1}^{1} (x+\xi_1+1)\left(\xi\xi_1 + \frac{1}{3}\right) d\xi_1$$

$$= -\frac{8}{3}(x+\xi+1) + 4\int_{-1}^{1}\left[x\xi\xi_1 + \frac{1}{3}x + \xi\xi_1^2 + \frac{1}{3}\xi_1 + \xi\xi_1 + \frac{1}{3}\right] d\xi_1$$

$$= -\frac{8}{3}(x+\xi+1) + \frac{4x\xi}{2}\left[\xi_1^2\right]_{-1}^{1} + \frac{4}{3}x\left[\xi_1\right]_{-1}^{1} + \frac{4\xi}{3}\left[\xi_1^3\right]_{-1}^{1}$$
$$+ \frac{4}{6}\left[\xi_1^2\right]_{-1}^{1} + \frac{4\xi}{2}\left[\xi_1^2\right]_{-1}^{1} + \frac{4}{3}\left[\xi_1\right]_{-1}^{1}$$

$$= -\frac{8}{3}(x+\xi+1) + 0 + \frac{4}{3}x(2) + \frac{8}{3}\xi + 0 + 0 + \frac{8}{3}$$

$$B_2(x,\xi) = -\frac{8}{3}(x+\xi+1) + \frac{8}{3}(x+\xi+1) = 0$$

Now $\quad C_3 = \int_{-1}^{1} B_2(\xi_1,\xi_1)\, d\xi_1 = \int_{-1}^{1} 0 \cdot d\xi_1 = 0$

$\Rightarrow C_3 = 0$ $\quad$ Similarly $B_3 = 0$, $C_3 = 0$, $B_4 = 0$, $C_4 = 0$ etc

$\because D(x,\xi;\lambda) = K(x,\xi) - \lambda . B_1(x,\xi)$

$[\because B_2(x,\xi) = 0 = B_3(x,\xi) = \cdots$

$= x+\xi+1 - \lambda(-2)[x\xi + \frac{1}{3}]$

$D(x,\xi;\lambda) = x+\xi+1+2\lambda(x\xi+\frac{1}{3})$

$D(\lambda) = 1 - \lambda C_1 + \frac{\lambda^2}{\lfloor 2}C_2 + 0 + 0 + \cdots$

$[\text{as } C_3 = 0 = C_4 = \cdots]$

$= 1 - \lambda(2) + \frac{\lambda^2}{\lfloor 2}\left(-\frac{8}{3}\right)$

$D(\lambda) = 1 - 2\lambda - \frac{8}{6}\lambda^2 = \left(1 - 2\lambda - \frac{4}{3}\lambda^2\right)$

$\therefore R(x,\xi;\lambda) = \frac{D(x,\xi;\lambda)}{D(\lambda)} = \frac{x+\xi+1+2\lambda(x\xi+\frac{1}{3})}{\left(1-2\lambda-\frac{4}{3}\lambda^2\right)}.$

Q9. Find resolvent kernel of

$K(x,\xi) = 4x\xi - x^2, \quad 0 \le x \le 1, \; 0 \le \xi \le 1$

Solution. Given $K(x,\xi) = B_0(x,\xi) = 4x\xi - x^2$

$, a = 0, b = 1$

$$B_1(x,\xi) = \int_0^1 K\begin{pmatrix} x & \xi_1 \\ \xi & \xi_1 \end{pmatrix} d\xi_1 = \int_0^1 \begin{vmatrix} K(x,\xi) & K(x,\xi_1) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) \end{vmatrix} d\xi_1$$

$$B_1(x,\xi) = \int_0^1 \begin{vmatrix} 4x\xi - x^2 & 4x\xi_1 - x^2 \\ 4\xi_1\xi - \xi_1^2 & 4\xi_1\xi_1 - \xi_1^2 \end{vmatrix} d\xi_1$$

$$= \int_0^1 \begin{vmatrix} 4x\xi - x^2 & 4x(\xi_1 - \xi) \\ 4\xi_1\xi - \xi_1^2 & 4\xi_1(\xi_1 - \xi) \end{vmatrix} d\xi_1$$

$$B_1(x,\xi) = 4\int_0^1 \begin{vmatrix} 4x\xi - x^2 & x \\ 4\xi_1\xi - \xi_1^2 & \xi_1 \end{vmatrix} . (\xi_1 - \xi)\, d\xi_1$$

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$$B_1(x,\xi) = 4\int_0^1 (\cancel{4x\xi\xi_1} - x^2\xi_1 - \cancel{4x\xi\xi_1} + x\xi_1^2)(\xi_1-\xi)\,d\xi_1$$

$$= 4\int_0^1 x\xi_1(\xi_1 - x)(\xi_1-\xi)\,d\xi_1$$

$$= 4x\int_0^1 \xi_1[\xi_1^2 - \xi_1\xi - x\xi_1 + x\xi]\,d\xi_1$$

$$= 4x\int_0^1 (\xi_1^3 - \xi_1^2\xi - x\xi_1^2 + x\xi\xi_1)\,d\xi_1$$

$$= 4x\left[\frac{1}{4} - \frac{1}{3}\xi - \frac{x}{3} + \frac{x\xi}{2}\right]$$

$$B_1(x,\xi) = x - \frac{4}{3}\xi x - \frac{4}{3}x^2 + 2x^2\xi$$

$$B_1(x,\xi) = \left(2x^2\xi - \frac{4}{3}x^2 + x - \frac{4}{3}x\xi\right)$$

$$c_1 = \int_0^1 B_0(\xi_1,\xi_1)\,d\xi_1 = \int_0^1 K(\xi_1,\xi_1)\,d\xi_1$$

$$= \int_0^1 (4\xi_1\xi_1 - \xi_1^2)\,d\xi_1 = \int_0^1 3\xi_1^2\,d\xi_1 = \frac{3}{3}(\xi_1^3)_0^1 = 1$$

$$\Rightarrow \boxed{c_1 = 1}$$

For $B_2(x,\xi)$, formula is

$$B_m(x,\xi) = c_m K(x,\xi) - m\int_0^1 K(x,\xi_1)\,B_{m-1}(\xi_1,\xi)\,d\xi_1$$

put $m=2$

$$B_2(x,\xi) = c_2 K(x,\xi) - 2\int_0^1 K(x,\xi_1)\,B_1(\xi_1,\xi)\,d\xi_1$$

first we find

$$c_2 = \int_0^1 B_1(\xi_1,\xi_1)\,d\xi_1 \quad \left\{ B_1(x,\xi) = 2x^2\xi - \frac{4}{3}x^2 + x - \frac{4}{3}x\xi \right]$$

$$= \int_0^1 \left(2\xi_1^2\xi_1 - \frac{4}{3}\xi_1^2 + \xi_1 - \frac{4}{3}\xi_1^2\right)d\xi_1$$

$$= \int_0^1 \left(2\xi_1^3 - \frac{4}{3}\xi_1^2 + \xi_1 - \frac{4}{3}\xi_1^2\right)d\xi_1 = \frac{2}{4} - \frac{4}{9} + \frac{1}{2} - \frac{4}{9}$$

$$c_2 = 1 - \frac{8}{9} = \frac{1}{9} \Rightarrow c_2 = \frac{1}{9}$$

$$\therefore B_2(x,\xi) = \frac{1}{9}K(x,\xi) - 2\int_0^1 K(x,\xi_1)\,B_1(\xi_1,\xi)\,d\xi_1$$

$$K(x,\xi) = 4x\xi - x^2 \Rightarrow K(x,\xi_1) = 4x\xi_1 - x^2$$

$$B_1(x,\xi) = 2x^2\xi - \frac{4}{3}x^2 + x - \frac{4}{3}x\xi$$

$$B_1(\xi_1,\xi) = \left(2\xi_1^2\xi - \frac{4}{3}\xi_1^2 + \xi_1 - \frac{4}{3}\xi_1\xi\right)$$

$$\therefore B_2(x,\xi) = \frac{1}{9}(4x\xi - x^2) - 2\int_0^1 (4x\xi_1 - x^2)\left(2\xi_1^2\xi - \frac{4}{3}\xi_1^2 + \xi_1 - \frac{4}{3}\xi\xi_1\right) \times d\xi_1$$

$$= \frac{1}{9}(4x\xi - x^2) - I_1$$

where

$$I_1 = \int_0^1 \Big[8x\xi_1\left(2\xi_1^2\xi - \frac{4}{3}\xi_1^2 + \xi_1 - \frac{4}{3}\xi\xi_1\right) - 2x^2\left(2\xi_1^2\xi - \frac{4}{3}\xi_1^2 + \xi_1 - \frac{4}{3}\xi\xi_1\right)\Big]\,d\xi_1$$

$$= \int_0^1 \Big[16x\xi_1^3\xi - \frac{32}{3}x\xi_1^3 + 8x\xi_1^2 - \frac{32}{3}x\xi\xi_1^2 - 4x^2\xi_1^2\xi + \frac{8}{3}x^2\xi_1^2 - 2x^2\xi_1 + \frac{8}{3}x^2\xi\xi_1\Big]\,d\xi_1$$

$$= 16x\xi\cdot\frac{1}{4} - \frac{32}{3}x\cdot\frac{1}{4} + \frac{8}{3}x - \frac{32}{3}x\xi\cdot\frac{1}{3} - 4x^2\xi\cdot\frac{1}{3} + \frac{8}{3}x^2\cdot\frac{1}{3} - x^2 + \frac{8}{3}x^2\xi\cdot\frac{1}{2}$$

$$= \underline{4x\xi} - \cancel{\frac{8}{3}x} + \cancel{\frac{8}{3}x} - \frac{32}{9}\underline{x\xi} - \underline{\frac{4}{3}x^2\xi} + \frac{8}{9}x^2 - x^2 + \underline{\frac{4}{3}x^2\xi}$$

$$= x\xi\left[4 - \frac{32}{9}\right] - \frac{1}{9}x^2 = \frac{4}{9}x\xi - \frac{1}{9}x^2$$

$$I_1 = \frac{1}{9}(4x\xi - x^2) \Rightarrow B_2(x,\xi) = \frac{1}{9}(4x\xi - x^2) - \frac{1}{9}(4x\xi - x^2)$$

$$\Rightarrow B_2(x,\xi) = 0.$$

$$C_2 = \int_0^1 B_2(\xi_1, \xi_1)\, d\xi_1 = 0 \text{ as } B_2 = 0.$$

$$\Rightarrow C_3 = 0 = C_4 = \cdots$$

$$B_3 = 0 = B_4 = \cdots$$

$$\therefore D(x, \xi; \lambda) = B_0(x,\xi) - \frac{\lambda}{\lfloor 1} B_1(x,\xi) + \frac{\lambda^2}{\lfloor 2} B_2(x,\xi) - \cdots$$

$$D(x, \xi; \lambda) = (4x\xi - x^2) - \lambda\left[2x^2\xi - \frac{4}{3}x^2 + x - \frac{4}{3}x\xi\right]$$

$[\because B_2(x,\xi) = 0 = B_3(x,\xi) \cdots$

$$D(\lambda) = 1 - \frac{\lambda}{\lfloor 1} C_1 + \frac{\lambda^2}{\lfloor 2} C_2 - \frac{\lambda^3}{\lfloor 3} C_3 + \cdots$$

$$D(\lambda) = 1 - \lambda \cdot 1 + \frac{\lambda^2}{\lfloor 2} + 0 \quad \{\because C_3 = 0 = C_4 = \cdots\}$$

$[\because C_1 = 1, C_2 = \frac{1}{9}]$

$$\therefore R(x, \xi; \lambda) = \frac{D(x, \xi; \lambda)}{D(\lambda)} = \frac{(4x\xi - x^2) - \lambda\left[2x^2\xi - \frac{4}{3}x^2 + x - \frac{4}{3}x\xi\right]}{\left[1 - \frac{\lambda}{9} + \frac{\lambda^2}{18}\right]}$$

Question for Practice

Find the resolvent Kernel of the following:

(1) $K(x, \xi) = 1, \ a = 0, b = 1$

Ans $R(x, \xi; \lambda) = \left(\frac{1}{1-\lambda}\right)$

(2) $K(x, \xi) = x\xi, \quad a = 0, \ b = 10$

$$R(x, \xi; \lambda) = \frac{x\xi}{\left(1 - \frac{1000}{3}\lambda\right)}$$

(3) $K(x, \xi) = 2e^x . e^\xi, a = 0, b = 1$, Ans $R(x, \xi; \lambda) = \frac{2e^x . e^\xi}{[1 - (e^2 - 1)\lambda]}$

(4) $K(x, \xi) = x, \ a = 4, \ b = 10$

Ans. $R(x, \xi; \lambda) = \left[\frac{x}{1 - 42\lambda}\right]$