SUBJECT: MATHEMATICS

PAPER: ADVANCED MATHEMATICAL METHODS
PAPER CODE:H 3051(PAPER III)
CLASS: MSc III SEMESTER
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To find the solution of

$$\phi(x) = F(x) + \lambda \int_a^b K(x,\xi)\,\phi(\xi)\,d\xi. \quad — ①$$

We know that the solution of ① is

$$\phi(x) = F(x) + \lambda \int_a^b R(x,\xi;\lambda)\,F(\xi)\,d\xi$$

where

$$R(x,\xi;\lambda) = \frac{D(x,\xi;\lambda)}{D(\lambda)}, \quad D(\lambda) \neq 0$$

$$D(x,\xi;\lambda) = K(x,\xi) + \sum \frac{(-\lambda)^m}{\lfloor m} \int_a^b\int_a^b \cdots \int_a^b K\begin{pmatrix} x & \xi_1 & \xi_2 & \cdots & \xi_m \\ \xi & \xi_1 & \xi_2 & & \xi_m \end{pmatrix} d\xi_1\, d\xi_2 \cdots d\xi_m$$

$$D(\lambda) = 1 + \sum \frac{(-\lambda)^m}{\lfloor m} \int_a^b\int_a^b \cdots \int_a^b K\begin{pmatrix} \xi_1 & \xi_2 & \cdots & \xi_m \\ \xi_1 & \xi_2 & \cdots & \xi_m \end{pmatrix} d\xi_1\, d\xi_2 \cdots d\xi_m$$

$$K\begin{pmatrix} x & \xi_1 & \xi_2 & \cdots & \xi_m \\ \xi & \xi_1 & \xi_2 & & \xi_m \end{pmatrix} = \begin{vmatrix} K(x,\xi) & K(x,\xi_1) & K(x,\xi_2) & \cdots & K(x,\xi_m) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) & K(\xi_1,\xi_2) & \cdots & K(\xi_1,\xi_m) \\ \vdots & & & & \\ K(\xi_m,\xi) & K(\xi_m,\xi_1) & K(\xi_m,\xi_2) & \cdots & K(\xi_m,\xi_m) \end{vmatrix}$$

$$K\begin{pmatrix} \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_m \\ \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_m \end{pmatrix}$$

$$= \begin{vmatrix} K(\xi_1, \xi_1) & K(\xi_1, \xi_2) & K(\xi_1, \xi_3) & \cdots & K(\xi_1, \xi_m) \\ K(\xi_2, \xi_1) & K(\xi_2, \xi_2) & K(\xi_2, \xi_3) & \cdots & K(\xi_2, \xi_m) \\ \vdots & & & & \\ K(\xi_m, \xi_1) & K(\xi_m, \xi_2) & K(\xi_m, \xi_3) & \cdots & K(\xi_m, \xi_m) \end{vmatrix}$$

Note. Here $D(\lambda)$ is called Fredholm determinants

$R(x, \xi; \lambda)$ is called resolvent kernel.

Also

$$D(x, \xi; \lambda) = B_0(x, \xi) + \sum \frac{(-1)^m \lambda^m}{\lfloor m} B_m(x, \xi)$$

where $B_0(x, \xi) = K(x, \xi)$

$$B_1(x, \xi) = \int_a^b K\begin{pmatrix} x & \xi_1 \\ \xi & \xi_1 \end{pmatrix} d\xi_1$$

$$= \int_a^b \begin{vmatrix} K(x, \xi) & K(x, \xi_1) \\ K(\xi_1, \xi) & K(\xi_1, \xi_1) \end{vmatrix} d\xi_1$$

$$B_2(x, \xi) = \int_a^b \int_a^b K\begin{pmatrix} x & \xi_1 & \xi_2 \\ \xi & \xi_1 & \xi_2 \end{pmatrix} d\xi_1 \, d\xi_2$$

$$B_2(x, \xi) = \int_a^b \int_a^b \begin{vmatrix} K(x, \xi) & K(x, \xi_1) & K(x, \xi_2) \\ K(\xi_1, \xi) & K(\xi_1, \xi_1) & K(\xi_1, \xi_2) \\ K(\xi_2, \xi) & K(\xi_2, \xi_1) & K(\xi_2, \xi_2) \end{vmatrix} d\xi_1 \, d\xi_2$$

$$B_m(x,\xi) = \int_a^b\int_a^b \cdots \int_a^b K\begin{pmatrix} x & \xi_1 & \xi_2 & \cdots & \xi_m \\ \xi & \xi_1 & \xi_2 & & \xi_m \end{pmatrix} d\xi_1\, d\xi_2 \cdots d\xi_m$$

$$c_1 = \int_a^b K(\xi_1, \xi_1)\, d\xi_1$$

$$c_2 = \int_a^b\int_a^b K\begin{pmatrix} \xi_1 & \xi_2 \\ \xi_1 & \xi_2 \end{pmatrix} d\xi_1\, d\xi_2 = \int\int \begin{vmatrix} K(\xi_1, \xi_1) & K(\xi_1, \xi_2) \\ K(\xi_2, \xi_1) & K(\xi_2, \xi_2) \end{vmatrix} d\xi_1\, d\xi_2$$

$\vdots$

$$c_m = \int_a^b\int_a^b \cdots \int_a^b K\begin{pmatrix} \xi_1 & \xi_2 & \cdots & \xi_m \\ \xi_1 & \xi_2 & & \xi_m \end{pmatrix} d\xi_1\, d\xi_2 \cdots d\xi_m$$

$$= \int_a^b\int_a^b \cdots \int_a^b \begin{vmatrix} K(\xi_1, \xi_1), & K(\xi_1, \xi_2) & \cdots & , K(\xi_1, \xi_m) \\ K(\xi_2, \xi_1) & K(\xi_2, \xi_2) & \cdots & K(\xi_2, \xi_m) \\ \vdots & & & \\ K(\xi_m, \xi_1) & K(\xi_m, \xi_2) & \cdots & K(\xi_m, \xi_m) \end{vmatrix} d\xi_1\, d\xi_2 \cdots d\xi_m.$$

Relation between $B_m(x, \xi)$ and c_m

$$B_m(x, \xi) = c_m K(x, \xi) - m \int_a^b K(x, \xi_1) B_{m-1}(\xi_1, \xi)\, d\xi_1$$

i.e.

$$B_1(x, \xi) = c_1 K(x, \xi) - 1 \cdot \int_a^b K(x, \xi_1) B_0(\xi_1, \xi)\, d\xi_1$$

$$B_2(x, \xi) = c_2 K(x, \xi) - 2\int_a^b K(x, \xi_1) B_1(\xi_1, \xi)\, d\xi_1$$

$$B_3(x, \xi) = c_3 K(x, \xi) - 3\int_a^b K(x, \xi_1) B_2(\xi_1, \xi)\, d\xi_1$$

$$B_4(x, \xi) = c_4 K(x, \xi) - 4\int_a^b K(x, \xi_1) B_3(\xi_1, \xi)\, d\xi_1$$

Find $D(\lambda)$ (Fredholm Determinants) and $R(x, \xi; \lambda)$ (Resolvent Kernel of the following

$$K(x, \xi) = x e^{\xi}, \quad a = 0, \; b = 1$$

Solution. We know

$$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{\lfloor m} C_m$$

$$R(x, \xi; \lambda) = \frac{D(x, \xi; \lambda)}{D(\lambda)}$$

$$, \; D(x, \xi; \lambda) = B_0(x, \xi) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{\lfloor m} B_m(x, \xi)$$

$$B_0(x, \xi) = K(x, \xi) = x e^{\xi}$$

$$B_1(x, \xi) = \int_0^1 K\begin{pmatrix} x & \xi_1 \\ \xi & \xi_1 \end{pmatrix} d\xi_1 = \int_0^1 \begin{vmatrix} K(x, \xi) & K(x, \xi_1) \\ K(\xi_1, \xi) & K(\xi_1, \xi_1) \end{vmatrix} d\xi_1$$

$$= \int_0^1 \begin{vmatrix} x e^{\xi} & x e^{\xi_1} \\ \xi_1 e^{\xi} & \xi_1 e^{\xi_1} \end{vmatrix} d\xi_1 = \int_0^1 \left(x \xi_1 e^{\xi + \xi_1} - x \xi_1 e^{\xi + \xi_1} \right) d\xi_1$$

$$= 0.$$

$$B_2(x, \xi) = \int_0^1 \int_0^1 K\begin{pmatrix} x & \xi_1 & \xi_2 \\ \xi & \xi_1 & \xi_2 \end{pmatrix} d\xi_1 d\xi_2 = \int_0^1 \int_0^1 \begin{vmatrix} K(x, \xi) & K(x, \xi_1) & K(x, \xi_2) \\ K(\xi_1, \xi) & K(\xi_1, \xi_1) & K(\xi_1, \xi_2) \\ K(\xi_2, \xi) & K(\xi_2, \xi_1) & K(\xi_2, \xi_2) \end{vmatrix} \cdot d\xi_1 d\xi_2$$

$$B_2(x, \xi) = \int_0^1 \int_0^1 \begin{vmatrix} x e^{\xi} & x e^{\xi_1} & x e^{\xi_2} \\ \xi_1 e^{\xi} & \xi_1 e^{\xi_1} & \xi_1 e^{\xi_2} \\ \xi_2 e^{\xi} & \xi_2 e^{\xi_1} & \xi_2 e^{\xi_2} \end{vmatrix} d\xi_1 d\xi_2$$

$$B_2(\eta,\xi) = \int_0^1\int_0^1 e^{\xi+\xi_1+\xi_2} \begin{vmatrix} x & x & x \\ \xi_1 & \xi_1 & \xi_1 \\ \xi_2 & \xi_2 & \xi_2 \end{vmatrix} d\xi_1\, d\xi_2$$

$$= \int_0^1\int_0^1 e^{\xi+\xi_1+\xi_2} \cdot D_3\, d\xi_1\, d\xi_2$$

$$D_3 = \begin{vmatrix} x & x & x \\ \xi_1 & \xi_1 & \xi_1 \\ \xi_2 & \xi_2 & \xi_2 \end{vmatrix} = \begin{vmatrix} x & 0 & 0 \\ \xi_1 & 0 & 0 \\ \xi_2 & 0 & 0 \end{vmatrix}, \quad \begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{matrix}$$

$$= 0.$$

$$\therefore B_2(\eta,\xi) = \int_0^1\int_0^1 0 \cdot d\xi_1\, d\xi_2 = 0.$$

$$C_1 = \int_0^1 K(\xi_1,\xi_1)\, d\xi_1 = \int_0^1 \xi_1 e^{\xi_1}\, d\xi_1$$

$$= \xi_1 \cdot e^{\xi_1}\Big|_0^1 - \int_0^1 1 \cdot e^{\xi_1}\, d\xi_1$$

$$C_1 = (1\cdot e - 0) - (e^{\xi_1})_0^1 = e - (e-1) = 1$$

$$C_2 = \int_0^1\int_0^1 K\begin{pmatrix} \xi_1 & \xi_2 \\ \xi_1 & \xi_2 \end{pmatrix} d\xi_1\, d\xi_2$$

$$C_2 = \int_0^1\int_0^1 \begin{vmatrix} K(\xi_1\ \xi_1) & K(\xi_1\ \xi_2) \\ K(\xi_2\ \xi_1) & K(\xi_2\ \xi_2) \end{vmatrix} d\xi_1\, d\xi_2$$

$$C_2 = \int_0^1\int_0^1 \begin{vmatrix} \xi_1 e^{\xi_1} & \xi_1 e^{\xi_2} \\ \xi_2 e^{\xi_1} & \xi_2 e^{\xi_2} \end{vmatrix} d\xi_1\, d\xi_2$$

$$C_2 = \int_0^1\int_0^1 \left(\xi_1 \xi_2 e^{\xi_1+\xi_2} - \xi_1 \xi_2 e^{\xi_1+\xi_2}\right) d\xi_1\, d\xi_2 = 0.$$

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$\because$ $D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{\lfloor m} c_m$

$\Rightarrow D(\lambda) = 1 - \frac{\lambda}{\lfloor 1} c_1 + \frac{\lambda^2}{\lfloor 2} c_2 - \frac{\lambda^3}{\lfloor 3} c_3 + \cdots$

$\Rightarrow D(\lambda) = (1-\lambda)$, $c_2 = 0 = c_3 = \cdots$

$D(x,\xi;\lambda) = B_0(x,\xi) + \sum \frac{(-\lambda)^m}{\lfloor m} B_m(x,\xi)$

$= K(x,\xi) - \frac{\lambda}{\lfloor 1} B_1(x,\xi) + \frac{\lambda^2}{\lfloor 2} B_2(x,\xi) - \frac{\lambda^3}{\lfloor 3} B_3(x,\xi) + \cdots$

$= xe^{\xi} - \lambda \cdot 0 + 0 + 0$

$D(x,\xi;\lambda) = xe^{\xi}$

$\because$ $R(x,\xi;\lambda) = \frac{D(x,\xi;\lambda)}{D(\lambda)} = \frac{xe^{\xi}}{(1-\lambda)}, \lambda \neq 1$

$$\boxed{\begin{aligned} &\because \quad D(\lambda) = (1-\lambda) \\ &R(x,\xi;\lambda) = \frac{xe^{\xi}}{(1-\lambda)} \end{aligned}}$$

(2.) Given $K(x,\xi) = 2x-\xi$, $0 \le x \le 1$, $0 \le \xi \le 1$

Find $D(\lambda)$ and $R(x,\xi;\lambda) = ?$

Solution. Given

$K(x,\xi) = 2x-\xi$, $a=0$, $b=1$ —— (1)

To find $D(\lambda)$ and $R(x,\xi;\lambda)$.

We know

$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{\lfloor m} c_m$ —— (2)

$R(x,\xi;\lambda) = \frac{D(x,\xi;\lambda)}{D(\lambda)}$, $D(\lambda) \neq 0$ —— (3)

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$$D(x,\xi;\lambda) = B_0(x,\xi) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{\lfloor m} B_m(x,\xi)$$

$$= K(x,\xi) - \frac{\lambda}{\lfloor 1} B_1(x,\xi) + \frac{\lambda^2}{\lfloor 2} B_2(x,\xi) - \frac{\lambda^3}{\lfloor 3} B_3(x,\xi)$$

$$+ \frac{\lambda^4}{\lfloor 4} B_4(x,\xi) - \cdots \quad (4)$$

$$, \quad B_0(x,\xi) = K(x,\xi)$$

$$B_1(x,\xi) = \int_0^1 K\begin{pmatrix} x & \xi_1 \\ \xi & \xi_1 \end{pmatrix} d\xi_1 = \int_0^1 \begin{vmatrix} K(x,\xi) & K(x,\xi_1) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) \end{vmatrix} d\xi_1$$

$$B_1(x,\xi) = \int_0^1 \begin{vmatrix} 2x-\xi & 2x-\xi_1 \\ 2\xi_1-\xi & 2\xi_1-\xi_1 \end{vmatrix} d\xi_1 = \int_0^1 \begin{vmatrix} 2x-\xi & \xi-\xi_1 \\ 2\xi_1-\xi & \xi-\xi_1 \end{vmatrix} d\xi_1 \quad C_2 \to C_2 - C_1$$

$$B_1(x,\xi) = \int_0^1 (\xi-\xi_1) \begin{vmatrix} 2x-\xi & 1 \\ 2\xi_1-\xi & 1 \end{vmatrix} d\xi_1 = \int_0^1 (\xi-\xi_1)(2x-\cancel{\xi}-2\xi_1+\cancel{\xi})\, d\xi_1$$

$$B_1(x,\xi) = \int_0^1 2(\xi-\xi_1)(x-\xi_1)\, d\xi_1 = 2\int_0^1 (x\xi - \xi\xi_1 - x\xi_1 + \xi_1^2)\, d\xi_1$$

$$B_1(x,\xi) = 2\left[x\xi - \frac{\xi}{2} - \frac{x}{2} + \frac{1}{3}\right] = 2x\xi - \xi - x - \frac{2}{3}$$

$$B_1(x,\xi) = -\left[x+\xi - 2\xi x - \frac{2}{3}\right]$$

$$B_2 = \int_0^1\int_0^1 K\begin{pmatrix} x & \xi_1 & \xi_2 \\ \xi & \xi_1 & \xi_2 \end{pmatrix} d\xi_1\, d\xi_2$$

$$B_2 = \int_0^1\int_0^1 \begin{vmatrix} K(x,\xi) & K(x,\xi_1) & K(x,\xi_2) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) & K(\xi_1,\xi_2) \\ K(\xi_2,\xi) & K(\xi_2,\xi_1) & K(\xi_2,\xi_2) \end{vmatrix} d\xi_1\, d\xi_2$$

$$B_2 = \int_0^1\int_0^1 \begin{vmatrix} 2x-\xi & 2x-\xi_1 & 2x-\xi_2 \\ 2\xi_1-\xi & 2\xi_1-\xi_1 & 2\xi_1-\xi_2 \\ 2\xi_2-\xi & 2\xi_2-\xi_1 & 2\xi_2-\xi_2 \end{vmatrix} d\xi_1\, d\xi_2$$

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$$B_2(x,\xi) = \int_0^1\int_0^1 \begin{vmatrix} 2x-\xi & \xi-\xi_1 & \xi-\xi_2 \\ 2\xi_1-\xi & \xi-\xi_1 & \xi-\xi_2 \\ 2\xi_2-\xi & \xi-\xi_1 & \xi-\xi_2 \end{vmatrix} d\xi_1\, d\xi_2 \quad , \begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{matrix}$$

$$= \int_0^1\int_0^1 (\xi-\xi_1)(\xi-\xi_2) \begin{vmatrix} 2x-\xi & 1 & 1 \\ 2\xi_1-\xi & 1 & 1 \\ 2\xi_2-\xi & 1 & 1 \end{vmatrix} d\xi_1\, d\xi_2$$

$B_2(x,\xi) = 0$ as two columns are equal.

Similarly $B_3(x,\xi_1) = 0 = B_4(x,\xi) = \cdots$

Now,

$$c_1 = \int_0^1 K(\xi_1,\xi_1)\, d\xi_1 = \int_0^1 (2\xi_1-\xi_1)\, d\xi_1 = \int_0^1 \xi_1\, d\xi_1 = \frac{1}{2}$$

$$c_2 = \int_0^1\int_0^1 K\begin{pmatrix} \xi_1 & \xi_2 \\ \xi_1 & \xi_2 \end{pmatrix} d\xi_1\, d\xi_2 = \int_0^1\int_0^1 \begin{vmatrix} K(\xi_1,\xi_1) & K(\xi_1,\xi_2) \\ K(\xi_2,\xi_1) & K(\xi_2,\xi_2) \end{vmatrix} d\xi_1\, d\xi_2$$

$$c_2 = \int_0^1\int_0^1 \begin{vmatrix} 2\xi_1-\xi_1 & 2\xi_1-\xi_2 \\ 2\xi_2-\xi_1 & 2\xi_2-\xi_2 \end{vmatrix} d\xi_1\, d\xi_2 = \int_0^1\int_0^1 \begin{vmatrix} \xi_1 & 2\xi_1-\xi_2 \\ 2\xi_2-\xi_1 & \xi_2 \end{vmatrix} d\xi_1\, d\xi_2$$

$$c_2 = \int_0^1\int_0^1 \left[\xi_1\xi_2 - (2\xi_1-\xi_2)(2\xi_2-\xi_1)\right] d\xi_1\, d\xi_2$$

$$c_2 = \int_0^1\int_0^1 \left(\xi_1\xi_2 - 4\xi_1\xi_2 + 2\xi_1^2 + 2\xi_2^2 - \xi_1\xi_2\right) d\xi_1\, d\xi_2$$

$$c_2 = \int_0^1 \left(\cancel{\frac{\xi_2}{2}} - 2\xi_2 + \frac{2}{3} + 2\xi_2^2 - \cancel{\frac{\xi_2}{2}}\right) d\xi_2$$

$$c_2 = \int_0^1 \left(-2\xi_2 + \frac{2}{3} + 2\xi_2^2\right) d\xi_2 = -1 + \frac{2}{3} + \frac{2}{3} = \frac{1}{3}$$

$$c_3 = \int_0^1\int_0^1\int_0^1 K\begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \\ \xi_1 & \xi_2 & \xi_3 \end{pmatrix} d\xi_1\, d\xi_2\, d\xi_3$$

$$= \int_0^1\int_0^1\int_0^1 \begin{vmatrix} K(\xi_1,\xi_1) & K(\xi_1,\xi_2) & K(\xi_1,\xi_3) \\ K(\xi_2,\xi_1) & K(\xi_2,\xi_2) & K(\xi_2,\xi_3) \\ K(\xi_3,\xi_1) & K(\xi_3,\xi_2) & K(\xi_3,\xi_3) \end{vmatrix} d\xi_1\, d\xi_2\, d\xi_3$$

$$= \int_0^1\int_0^1\int_0^1 \begin{vmatrix} 2\xi_1-\xi_1 & 2\xi_1-\xi_2 & 2\xi_1-\xi_3 \\ 2\xi_2-\xi_1 & 2\xi_2-\xi_2 & 2\xi_2-\xi_3 \\ 2\xi_3-\xi_1 & 2\xi_3-\xi_2 & 2\xi_3-\xi_3 \end{vmatrix} d\xi_1\, d\xi_2\, d\xi_3$$

$$= \int_0^1\int_0^1\int_0^1 \begin{vmatrix} \xi_1 & \xi_1-\xi_2 & \xi_1-\xi_3 \\ 2\xi_2-\xi_1 & \xi_1-\xi_2 & \xi_1-\xi_3 \\ 2\xi_3-\xi_1 & \xi_1-\xi_2 & \xi_1-\xi_3 \end{vmatrix} d\xi_1\, d\xi_2\, d\xi_3 \quad \begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{matrix}$$

$$c_3 = \int_0^1\int_0^1\int_0^1 (\xi_1-\xi_2)(\xi_1-\xi_3) \begin{vmatrix} \xi_1 & 1 & 1 \\ 2\xi_2-\xi_1 & 1 & 1 \\ 2\xi_3-\xi_1 & 1 & 1 \end{vmatrix} d\xi_1\, d\xi_2\, d\xi_3$$

$= 0$, two columns are equal,

Similarly $c_4 = 0 = c_5 = \cdots$.

$\because$ $R(x,\xi;\lambda) = \dfrac{D(x,\xi;\lambda)}{D(\lambda)} \Rightarrow$

$$D(x,\xi;\lambda) = B_0(x,\xi) - \frac{\lambda}{\lfloor 1} B_1(x,\xi) + \frac{\lambda^2}{\lfloor 2} B_2(x,\xi) - \cdots$$

$$\boxed{D(x,\xi;\lambda) = 2x-\xi + \lambda\left(x+\xi-2\xi x-\frac{2}{3}\right)}$$

$$D(\lambda) = 1 - \lambda c_1 + \frac{\lambda^2}{\lfloor 2} c_2 - \frac{\lambda^3}{\lfloor 3} c_3 + \cdots$$

$$\boxed{D(\lambda) = 1 - \frac{\lambda}{2} + \frac{\lambda^2}{\lfloor 2}\left(\frac{1}{3}\right) - 0 = \left(1-\frac{\lambda}{2}+\frac{\lambda^2}{6}\right)}$$

(3). Find $R(x,\xi;\lambda)$ of the following
$K(x,\xi) = x^2\xi - x\xi^2, \quad 0 \le x \le 1, \; 0 \le \xi \le 1$

Solution. Given

$$K(x,\xi) = x^2\xi - x\xi^2, \quad a=0, \; b=1 \quad \text{——— (1)}$$

To find $R(x,\xi;\lambda) = ?$

We know that $R(x,\xi;\lambda) = \dfrac{D(x,\xi;\lambda)}{D(\lambda)}$

where $D(x,\xi;\lambda) = B_0(x,\xi) - \dfrac{\lambda}{\lfloor 1} B_1(x,\xi) + \dfrac{\lambda^2}{\lfloor 2} B_2(x,\xi) - \dfrac{\lambda^3}{\lfloor 3} B_3(x,\xi) + \cdots$

$$B_0(x,\xi) = K(x,\xi) = x^2\xi - x\xi^2$$

$$B_1(x,\xi) = \int_0^1 K\begin{pmatrix} x & \xi_1 \\ \xi & \xi_1 \end{pmatrix} d\xi_1 = \int_0^1 \begin{vmatrix} K(x,\xi) & K(x,\xi_1) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) \end{vmatrix} d\xi_1$$

$$B_1(x,\xi) = \int_0^1 \begin{vmatrix} x^2\xi - x\xi^2 & x^2\xi_1 - x\xi_1^2 \\ \xi_1^2\xi - \xi_1\xi^2 & 0 \end{vmatrix} d\xi_1$$

$$B_1(x,\xi) = -\int_0^1 (\xi_1^2\xi - \xi_1\xi^2)(x^2\xi_1 - x\xi_1^2)\, d\xi_1$$

$$B_1(x,\xi) = -\int_0^1 \left[(\xi_1^2\xi - \xi_1\xi^2)(x^2\xi_1) - (\xi_1^2\xi - \xi_1\xi^2)(x\xi_1^2)\right] d\xi_1$$

$$B_1(x,\xi) = -\int_0^1 (x^2\xi\,\xi_1^3 - \xi^2\xi_1^2 x^2 - x\xi_1^4\xi + \xi_1^3\xi^2 x)\, d\xi$$

$$B_1(x,\xi) = -\left[\frac{x^2\xi}{4} - \frac{x^2\xi^2}{3} - \frac{x\xi}{5} + \frac{x\xi^2}{4}\right]$$

$$B_1(x,\xi) = -x\xi\left[\frac{x+\xi}{4} - \frac{x\xi}{3} - \frac{1}{5}\right]$$

$$B_2(x,\xi) = \int_0^1\int_0^1 K\begin{pmatrix} x & \xi_1 & \xi_2 \\ \xi & \xi_1 & \xi_2 \end{pmatrix} d\xi_1\, d\xi_2$$

$$= \int_0^1\int_0^1 \begin{vmatrix} K(x,\xi) & K(x,\xi_1) & K(x,\xi_2) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) & K(\xi_1,\xi_2) \\ K(\xi_2,\xi) & K(\xi_2,\xi_1) & K(\xi_2,\xi_2) \end{vmatrix} d\xi_1\, d\xi_2$$

$$= \int_0^1\int_0^1 \begin{vmatrix} x^2\xi - x\xi^2 & x^2\xi_1 - x\xi_1^2 & x^2\xi_2 - x\xi_2^2 \\ \xi_1^2\xi - \xi_1\xi^2 & \xi_1^2\xi_1 - \xi_1\xi_1^2 & \xi_1^2\xi_2 - \xi_1\xi_2^2 \\ \xi_2^2\xi - \xi_2\xi^2 & \xi_2^2\xi_1 - \xi_2\xi_1^2 & \xi_2^2\xi_2 - \xi_2\xi_2^2 \end{vmatrix} d\xi_1\, d\xi_2$$

$$= \int_0^1\int_0^1 x\,\xi_1\xi_2 \begin{vmatrix} \xi(x-\xi) & \xi_1(x-\xi_1) & \xi_2(x-\xi_2) \\ \xi(\xi_1-\xi) & \xi_1(\xi_1-\xi_1) & \xi_2(\xi_1-\xi_2) \\ \xi(\xi_2-\xi) & \xi_1(\xi_2-\xi_1) & \xi_2(\xi_2-\xi_2) \end{vmatrix} d\xi_1\, d\xi_2$$

$$= \int_0^1\int_0^1 x\,\xi\,\xi_1^2\xi_2^2 \begin{vmatrix} x-\xi & x-\xi_1 & x-\xi_2 \\ \xi_1-\xi & \xi_1-\xi_1 & \xi_1-\xi_2 \\ \xi_2-\xi & \xi_2-\xi_1 & \xi_2-\xi_2 \end{vmatrix} d\xi_1\, d\xi_2$$

$$= \int_0^1\int_0^x x\,\xi\,\xi_1^2\xi_2^2 \begin{vmatrix} x-\xi & \xi-\xi_1 & \xi-\xi_2 \\ \xi_1-\xi & \xi-\xi_1 & \xi-\xi_2 \\ \xi_2-\xi & \xi-\xi_1 & \xi-\xi_2 \end{vmatrix} d\xi_1\, d\xi_2$$

$C_2 \to C_2 - C_1$
$C_3 \to C_3 - C_1$

$$= \int_0^1\int_0^1 x\,\xi\,\xi_1^2\xi_2^2 (\xi-\xi_1)(\xi-\xi_2) \begin{vmatrix} x-\xi & 1 & 1 \\ \xi_1-\xi & 1 & 1 \\ \xi_2-\xi & 1 & 1 \end{vmatrix} d\xi_1\, d\xi_2$$

$= 0$ as two columns are same

Similarly $B_3(x,\xi) = 0 = B_4(x,\xi) = \cdots$.

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$$c_1 = \int_0^1 K(\xi_1,\xi_1)\,d\xi_1 = \int_0^1 (\xi_1^2\xi_1 - \xi_1\xi_1^2)\,d\xi_1 = 0.$$

$$c_2 = \int_0^1\int_0^1 K\begin{pmatrix}\xi_1 & \xi_2\\ \xi_1 & \xi_2\end{pmatrix} d\xi_1 d\xi_2 = \int_0^1\int_0^1 \begin{vmatrix} K(\xi_1,\xi_1) & K(\xi_1,\xi_2)\\ K(\xi_2,\xi_1) & K(\xi_2,\xi_2)\end{vmatrix} d\xi_1 d\xi_2$$

$$c_2 = \int_0^1\int_0^1 \begin{vmatrix} \xi_1^2\xi_1 - \xi_1\xi_1^2 & \xi_1^2\xi_2 - \xi_1\xi_2^2\\ \xi_2^2\xi_1 - \xi_2\xi_1^2 & \xi_2^2\xi_2 - \xi_2\xi_2^2\end{vmatrix} d\xi_1\, d\xi_2$$

$$c_2 = \int_0^1\int_0^1 \begin{vmatrix} 0 & \xi_1\xi_2(\xi_1-\xi_2)\\ \xi_1\xi_2(\xi_2-\xi_1) & 0\end{vmatrix} d\xi_1\, d\xi_2$$

$$c_2 = \int_0^1\int_0^1 +\ \xi_1^2\xi_2^2\,(\xi_1-\xi_2)^2\, d\xi_1\, d\xi_2$$

$$c_2 = \int_0^1\int_0^1 \xi_1^2\xi_2^2\,(\xi_1^2+\xi_2^2-2\xi_1\xi_2)\, d\xi_1\, d\xi_2$$

$$c_2 = \int_0^1\int_0^1 (\xi_1^4\xi_2^2 + \xi_1^2\xi_2^4 - 2\xi_1^3\xi_2^3)\, d\xi_1\, d\xi_2$$

$$c_2 = \int_0^1 \left(\tfrac{1}{5}\cdot\xi_2^2 + \tfrac{1}{3}\cdot\xi_2^4 - \tfrac{2}{4}\cdot\xi_2^3\right) d\xi_2$$

$$c_2 = \frac{1}{15} + \frac{1}{15} - \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{120}$$

$$c_3 = \int_0^1\int_0^1\int_0^1 K\begin{pmatrix}\xi_1 & \xi_2 & \xi_3\\ \xi_1 & \xi_2 & \xi_3\end{pmatrix} d\xi_1\, d\xi_2\, d\xi_3$$

$$c_3 = \int_0^1\int_0^1\int_0^1 \begin{vmatrix} K(\xi_1,\xi_1) & K(\xi_1,\xi_2) & K(\xi_1,\xi_3)\\ K(\xi_2,\xi_1) & K(\xi_2,\xi_2) & K(\xi_2,\xi_3)\\ K(\xi_3,\xi_1) & K(\xi_3,\xi_2) & K(\xi_3,\xi_3)\end{vmatrix} d\xi_1 d\xi_2 d\xi_3$$

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$$c_3 = \int_0^1\int_0^1\int_0^1 \begin{vmatrix} \xi_1^2\xi_1 - \xi_1\xi_1^2 & \xi_1^2\xi_2 - \xi_1\xi_2^2 & \xi_1^2\xi_3 - \xi_1\xi_3^2 \\ \xi_2^2\xi_1 - \xi_2\xi_1^2 & \xi_2^2\xi_2 - \xi_2\xi_2^2 & \xi_2^2\xi_3 - \xi_2\xi_3^2 \\ \xi_3^2\xi_1 - \xi_3\xi_1^2 & \xi_3^2\xi_2 - \xi_3\xi_2^2 & \xi_3^2\xi_3 - \xi_3\xi_3^2 \end{vmatrix} d\xi_1 d\xi_2 \cdot d\xi_3$$

$$c_3 = \int_0^1\int_0^1\int_0^1 \begin{vmatrix} \xi_1\xi_1(\xi_1-\xi_1) & \xi_1\xi_2(\xi_1-\xi_2) & \xi_1\xi_3(\xi_1-\xi_3) \\ \xi_2\xi_1(\xi_2-\xi_1) & \xi_2\xi_2(\xi_2-\xi_2) & \xi_2\xi_3(\xi_2-\xi_3) \\ \xi_3\xi_1(\xi_3-\xi_1) & \xi_3\xi_2(\xi_3-\xi_2) & \xi_3\xi_3(\xi_3-\xi_3) \end{vmatrix} d\xi_1 d\xi_2 d\xi_3$$

$$c_3 = \int_0^1\int_0^1\int_0^1 \xi_1\xi_2\xi_3 \begin{vmatrix} \xi_1(\xi_1-\xi_1) & \xi_1(\xi_1-\xi_2) & \xi_1(\xi_1-\xi_3) \\ \xi_2(\xi_2-\xi_1) & \xi_2(\xi_2-\xi_2) & \xi_2(\xi_2-\xi_3) \\ \xi_3(\xi_3-\xi_1) & \xi_3(\xi_3-\xi_2) & \xi_3(\xi_3-\xi_3) \end{vmatrix} d\xi_1 d\xi_2 d\xi_3$$

$$c_3 = \int_0^1\int_0^1\int_0^1 \xi_1^2\xi_2^2\xi_3^2 \begin{vmatrix} \xi_1-\xi_1 & \xi_1-\xi_2 & \xi_1-\xi_3 \\ \xi_2-\xi_1 & \xi_2-\xi_2 & \xi_2-\xi_3 \\ \xi_3-\xi_1 & \xi_3-\xi_2 & \xi_3-\xi_3 \end{vmatrix} d\xi_1 d\xi_2 d\xi_3$$

$$c_3 = \int_0^1\int_0^1\int_0^1 \xi_1^2\xi_2^2\xi_3^2 \begin{vmatrix} \xi_1-\xi_1 & \xi_1-\xi_2 & \xi_1-\xi_3 \\ \xi_2-\xi_1 & \xi_1-\xi_2 & \xi_1-\xi_3 \\ \xi_3-\xi_1 & \xi_1-\xi_2 & \xi_1-\xi_3 \end{vmatrix} d\xi_1 d\xi_2 d\xi_3$$

$C_2 \to C_2 - C_1$
$C_3 \to C_3 - C_1$

$$c_3 = \int_0^1\int_0^1\int_0^1 \xi_1^2\xi_2^2\xi_3^2(\xi_1-\xi_2)(\xi_1-\xi_3) \begin{vmatrix} 0 & 1 & 1 \\ \xi_2-\xi_1 & 1 & 1 \\ \xi_3-\xi_1 & 1 & 1 \end{vmatrix} d\xi_1 d\xi_2 d\xi_3$$

$c_3 = 0$ since two columns are equals

Similarly $c_4 = 0 = c_5 = \cdots$

$$\therefore R(x,\xi;\lambda) = \frac{D(x,\xi;\lambda)}{D(\lambda)}$$

$$= \frac{\left[K(x,\xi) - \frac{\lambda}{\lfloor 1}B_1(x,\xi) + \frac{\lambda^2}{\lfloor 2}B_2(x,\xi) - \frac{\lambda^3}{\lfloor 3}B_3(x,\xi) + \cdots\right]}{\left[1 - \frac{\lambda}{\lfloor 1}c_1 + \frac{\lambda^2}{\lfloor 2}c_2 - \frac{\lambda^3}{\lfloor 3}c_3 + \cdots\right]}$$

$$R(x,\xi;\lambda) = \frac{x^2\xi - x\xi^2 + \lambda x\xi\left(\frac{x+\xi}{4} - \frac{x\xi}{3} - \frac{1}{5}\right)}{1 - \frac{\lambda}{\lfloor 1}\cdot 0 + \frac{\lambda^2}{\lfloor 2}\cdot\frac{1}{120} + 0}$$

$$R(x,\xi;\lambda) = \frac{x\xi(x-\xi) + \lambda x\xi\left(\frac{x+\xi}{4} - \frac{x\xi}{3} - \frac{1}{5}\right)}{\left(1 + \frac{\lambda^2}{240}\right)}.$$

(4) Find $R(x,\xi;\lambda)$ of the following Kernel

$$K(x,\xi) = \sin x\cos\xi, \quad 0 \le x \le 2\pi, \quad 0 \le \xi \le 2\pi$$

Solution. Given $K(x,\xi) = \sin x\cos\xi$, $a=0$, $b=2\pi$

We know that

$$R(x,\xi;\lambda) = \frac{K(x,\xi) - \frac{\lambda}{\lfloor 1}B_1(x,\xi) + \frac{\lambda^2}{\lfloor 2}B_2(x,\xi) - \cdots}{1 - \frac{\lambda}{\lfloor 1}c_1 + \frac{\lambda^2}{\lfloor 2}c_2 - \frac{\lambda^3}{\lfloor 3}c_3 + \cdots} \quad \text{--- (1)}$$

Now, $$B_1(x,\xi) = \int_0^{2\pi} K\begin{pmatrix} x & \xi_1 \\ \xi & \xi_1 \end{pmatrix} d\xi_1 = \int_0^{2\pi} \begin{vmatrix} K(x,\xi) & K(x,\xi_1) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) \end{vmatrix} d\xi_1$$

$$B_1(x,\xi) = \int_0^{2\pi} \begin{vmatrix} \sin x\cos\xi & \sin x\cos\xi_1 \\ \sin\xi_1\cos\xi & \sin\xi_1\cos\xi_1 \end{vmatrix} d\xi_1 = \sin x\int_0^{2\pi}\begin{vmatrix} \cos\xi & \cos\xi_1 \\ \sin\xi_1\cos\xi & \sin\xi_1\cos\xi_1 \end{vmatrix} d\xi_1$$

$= 0$, Similarly $B_2(x,\xi) = 0 = B_3(x,\xi) = \cdots$

$$c_1 = \int_0^{2\pi} K(\xi_1,\xi_1)\,d\xi_1 = \int_0^{2\pi}\sin\xi_1\cos\xi_1\,d\xi_1 = 0$$

Similarly $c_2 = 0 = c_3 = \cdots$

So $$R(x,\xi;\lambda) = \frac{\sin x\cos\xi}{1}.$$