Pdf-06 - (01) H-3051 - CH-02

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Application of Fourier Series in initial and boundary value problem.

To solve wave equation $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2}\right)$ (1D)

using separation of variables.

Solution. We know that the partial differential equation of wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2}\right), \quad u = u(x,t) \text{ denotes the deflection of the wave.} \quad \text{---} (1)$$

Let $u(x,t) = X(x)\,T(t)$ ——— (2)

be the solution of (1) where $X(x)$ is the function of x only and $T(t)$ is the function of t only.

Differentiating (2) partially two times with respect to x & t respectively, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 X}{\partial x^2}\cdot T$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{d^2 X}{dx^2}\cdot T \quad \text{---} (3) \qquad [\because X \text{ is a function of } x \text{ only}]$$

and similarly

$$\frac{\partial^2 u}{\partial t^2} = X\cdot\frac{\partial^2 T}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = X\cdot\frac{d^2 T}{dt^2} \quad \text{---} (4) \qquad [\because T(t) \text{ is a function of } t \text{ only}]$$

from equation (1), (3) & (4), we have

$$X \frac{d^2T}{dt^2} = c^2 \cdot T \frac{d^2X}{dx^2}$$

$$\Rightarrow \quad \underset{(i)}{\frac{\frac{d^2X}{dx^2}}{X}} = \underset{(ii)}{\frac{\frac{d^2T}{dt^2}}{c^2T}} = \underset{(iii)}{k}, \text{ say} \quad \text{---} (5)$$

because these two subsidiary equations are independent.

Taking (i) and (iii), we have

$$\frac{d^2X}{dx^2} = kX \quad \text{———} (6)$$

Taking (ii) and (iii), we have

$$\frac{d^2T}{dt^2} = c^2k \cdot T \quad \text{———} (7)$$

Case I. $k = 0$

$$(6) \Rightarrow \frac{d^2X}{dx^2} = 0 \Rightarrow \frac{dX}{dx} = A_1 \Rightarrow X = A_1x + B_1 \quad \text{—} (8)$$

$$(7) \Rightarrow \frac{d^2T}{dt^2} = 0 \Rightarrow \frac{dT}{dt} = C_1 \Rightarrow T = C_1t + D_1 \quad \text{—} (9)$$

$$\therefore \quad u(x,t) = X(x)\,T(t) \Rightarrow$$

$$u(x,t) = (A_1x + B_1)(C_1t + D_1)$$

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Case (ii) $k > 0$ let $k = n^2$, say

(6) $\Rightarrow \frac{d^2X}{dx^2} = n^2 X \Rightarrow \left(\frac{d^2}{dx^2} - n^2\right) X = 0$

A.E is $D^2 - n^2 = 0 \Rightarrow D = \pm n$

$\Rightarrow X(x) = A_1 e^{-nx} + B_1 e^{nx}$

(7) $\Rightarrow \frac{d^2T}{dt^2} = c^2 n^2 T \Rightarrow (D^2 - c^2 n^2) T = 0$

A.E is $D^2 - c^2 n^2 = 0 \Rightarrow c = \pm cn$

$\therefore T = C_1 e^{-cnt} + D_1 e^{cnt}$

$\therefore$ Required solution is given by

$u(x, t) = X(x) T(t)$

$$\Rightarrow \boxed{u(x,t) = (A_1 e^{-nx} + B_1 e^{nx})(C_1 e^{-cnt} + D_1 e^{cnt})}$$

Case (iii) $k < 0$ let $k = -n^2$

(6) $\Rightarrow \frac{d^2X}{dx^2} = -n^2 X \Rightarrow (D^2 + n^2) X = 0$

A.E is $D^2 + n^2 = 0 \Rightarrow D = \pm in$

$\therefore X = A_1 \cos nx + B_1 \sin nx$

(7) $\Rightarrow \frac{d^2T}{dt^2} = -c^2 n^2 T \Rightarrow (D^2 + c^2 n^2) T = 0$

A.E is $D^2 + c^2 n^2 = 0 \Rightarrow D = \pm inc$

$\therefore T = (c_1 \cos nct + D_1 \sin nct)$

$\therefore$ Required solution is given by

$$\boxed{u(x,t) = (A_1 \cos nx + B_1 \sin nx)(C_1 \cos nct + D_1 \sin nct)}$$

Q.1. Solve the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

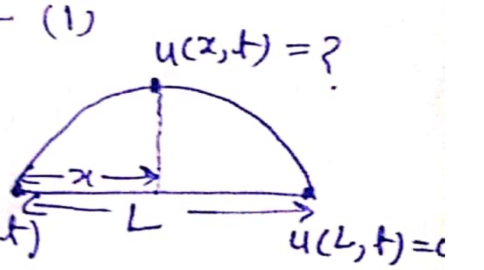
where $u(x,t) = P_0 \cos pt$ (P_0 is a constant)

when $x = L$ and $u = 0$ where $x = 0$

Solution. Note. Here $u = u(x,t)$.

Given wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{\partial^2 u}{\partial x^2}\right] \quad -(1)$$



given conditions are

$$u(0,t) = 0 \quad -(2)$$

$$u(L,t) = P_0 \cos pt \quad -(3)$$

We know the solution of (1) is given by

$$u(x,t) = (A_1 \cos nx + B_1 \sin nx)(C_1 \cos nct + D_1 \sin nct) \quad -(5)$$

Equation (2) and (5) $\Rightarrow$

$$0 = (A_1 + 0)(C_1 \cos nct + D_1 \sin nct)$$

$\Rightarrow A_1 = 0$ as $(C_1 \cos nct + D_1 \sin nct) \neq 0$.

$\therefore$ Equation (5) $\Rightarrow$

$$u(x,t) = B_1 \sin nx\,(C_1 \cos nct + D_1 \sin nct)$$

$$u(x,t) = B_1 C_1 \sin nx \cos nct + B_1 D_1 \sin nx \sin nct \quad -(6)$$

put $x = L$, we have

$$u(L,t) = B_1 C_1 \sin nL \cos nct + B_1 D_1 \sin nL \sin nct$$

Put $u(L,t) = P_0 \cos pt$, we have

$$P_0 \cos pt = B_1 C_1 \sin nL \cos nct + B_1 D_1 \sin nL \sin nct$$

On comparing, we have

$B_1 C_1 \sin nL = P_0$ and $p = nc$, $B_1 D_1 = 0$ — (7)

$\therefore \quad B_1 C_1 = \dfrac{P_0}{\sin nL} = \dfrac{P_0}{\sin \frac{Lp}{c}} \quad \left[\because n = \frac{p}{c}\right]$ — (8)

But $B_1 D_1 = 0$ & $B_1 C_1 = \dfrac{P_0}{\sin \frac{Lp}{c}}$

Putting these values in (6), we get

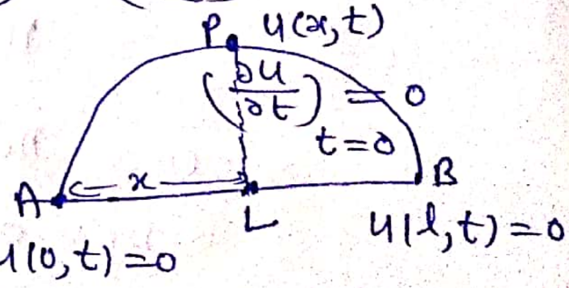
$$u(x,t) = \frac{P_0}{\sin \frac{Lp}{c}} \cdot \sin nx \cos nct + 0$$

$$\boxed{u(x,t) = \frac{P_0}{\sin \frac{Lp}{c}} \sin \frac{px}{c} \cos pt}$$

Required answer.

Q2. A string is stretched to two fixed points distance l apart. Motion is started by displacing the string in the form $u = a \sin\left(\frac{\pi x}{l}\right)$ from which it is released at time $t=0$. Show that the displacement at any point x from one end at time t is given by

$$u(x,t) = a \sin\left(\frac{\pi x}{l}\right) \cos\left(\frac{\pi c t}{l}\right).$$



Solution. The partial differential equation of the given string is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2}\right) \quad \text{— (1)}$$

Initial and boundary conditions are
(I.C) (b.C)

$$u(0,t) = 0 \quad \text{--- (2)} \quad (\text{b.c})$$

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0 \quad \text{--- (3)} \quad (\text{I.C})$$

$$u(l,t) = 0 \quad \text{--- (4)} \quad (\text{b.c})$$

$$u(x,0) = a\sin\left(\frac{\pi}{l}x\right) \quad \text{--- (5)} \quad (\text{I.C})$$

The solution of (1) is given by

$$u(x,t) = (A_1\cos nx + B_1\sin nx)(C_1\cos nct + D_1\sin nct) \quad \text{--- (6)}$$

Equation (2) and (6) implies

$$0 = (A_1 + 0)(C_1\cos nct + D_1\sin nct)$$

$$\Rightarrow A_1 = 0 \text{ as } (C_1\cos nt + D_1\sin nct) \neq 0$$

$\therefore$ Equation (6) $\Rightarrow$

$$u(x,t) = (B_1\sin nx)(C_1\cos nct + D_1\sin nct)$$

$$u(x,t) = B_1C_1\sin nx\cos nct + B_1D_1\sin nx\sin nct \quad \text{--- (7)}$$

$$\frac{\partial u}{\partial t} = -B_1C_1\sin nx \cdot nc\sin nct + B_1D_1\sin nx \cdot nc\cos nct$$

put $t=0$

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = ncB_1D_1\sin nx$$

$$\Rightarrow 0 = ncB_1D_1\sin nx \quad \text{, from (3)}$$

$$\Rightarrow B_1D_1 = 0 \text{ as } \sin nx \neq 0 \quad \text{--- (8)}$$

$\therefore$ Equation (7) and (8) $\Rightarrow$

$$u(x,t) = B_1C_1\sin nx\cos nct \quad \text{--- (9)}$$

put $x = l$

$$u(l,t) = B_1 C_1 \sin nl \cos nct \quad — (10)$$

$\Rightarrow$ $0 = B_1 C_1 \sin nl \cos nct$ from (4)

$\Rightarrow$ $B_1 C_1 = 0$ or $\sin nl = 0$ as $\cos nct \neq 0$

If $B_1 C_1 \neq 0$ then $\sin nl = 0 = \sin m\pi$

$\Rightarrow nl = m\pi \Rightarrow n = \frac{1}{l} m\pi \quad — (11)$

∴ (9) & (11) $\Rightarrow$

$$u(x,t) = B_1 C_1 \sin \frac{m\pi x}{l} \cos\left(\frac{m\pi ct}{l}\right) \quad — (12)$$

put $t = 0$

$$u(x,0) = B_1 C_1 \sin \frac{m\pi x}{l}$$

$$a \sin \frac{\pi x}{l} = B_1 C_1 \sin \frac{m\pi x}{l}$$

on comparing, we have

$$a = B_1 C_1$$

$$\frac{\pi x}{l} = \frac{m\pi x}{l} \Rightarrow m = 1 \quad —$$

So putting $B_1 C_1 = a$ & $m = 1$ in (12), we get

$$\boxed{u(x,t) = a \sin \frac{\pi x}{l} \cos\left(\frac{\pi ct}{l}\right)}$$

which is required solution.

Q.3. The vibrations of an elastic string is governed by the partial differential equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$. The length of the string is π and the ends are fixed. The initial velocity is zero and the initial deflection is $u(x,0) = 2(\sin x + \sin 3x)$. Find the deflection of the vibrating string for $t > 0$.

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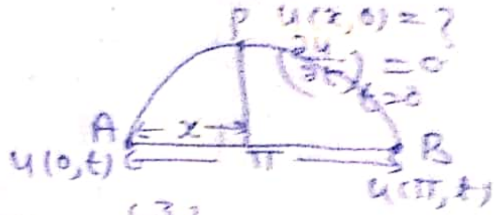
Solution. Given equation of the string is

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad (\text{Here } c=1) \quad \text{—— (1)}$$

Conditions are



$$u(0,t) = 0 \quad \text{—— (2)}$$

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0 \quad \text{—— (3)}$$

$$u(\pi, t) = 0 \quad \text{—— (4)}$$

$$u(x,0) = 2(\sin x + \sin 3x) \quad \text{—— (5)}$$

The solution of (1) is given by

$$u(x,t) = (A_1 \cos nx + B_1 \sin nx)(C_1 \cos nt + D_1 \sin nt) \quad \text{—— (6)}$$

put $x = 0$

$$u(0,t) = (A_1 + 0)(C_1 \cos nt + D_1 \sin nt)$$

$$\Rightarrow \quad 0 = A_1 (C_1 \cos nt + D_1 \sin nt)$$

$$\Rightarrow \quad A_1 = 0 \quad \text{as} \ (C_1 \cos nt + D_1 \sin nt) \neq 0$$

put $A_1 = 0$ in (6), we get

$$u(x,t) = B_1 \sin nx (C_1 \cos nt + D_1 \sin nt)$$

$$u(x,t) = B_1 C_1 \sin nx \cos nt + B_1 D_1 \sin nx \sin nt \quad \text{—— (7)}$$

$$\frac{\partial u}{\partial t} = -n B_1 C_1 \sin nx \sin t + n B_1 D_1 \sin nx \cdot \cos nt$$

put $t = 0$

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0 + n B_1 D_1 \sin nx \quad [\because \cos 0 = 1]$$

$$\Rightarrow \quad 0 = n B_1 D_1 \sin nx$$

$$\Rightarrow \quad B_1 D_1 = 0 \quad \text{as} \ \sin nx \neq 0$$

so $u(x,t) = B_1 C_1 \sin nx \cos nt$

put $x = \pi$

$$u(\pi, t) = B_1 C_1 \sin n\pi \cos nt$$

$$0 = B_1 C_1 \sin n\pi \cdot \cos nt$$

$$\Rightarrow \sin n\pi = 0 \text{ if } B_1 C_1 \neq 0$$

$\Rightarrow$ n must be an integer

Now, we have

$$u(x, t) = B_1 C_1 \sin nx \cos nt \text{ where } n \in I, \text{ an Integer}$$

put $t = 0$, we have

i.e. $u(x, 0) = 2(\sin x + \sin 3x) \Rightarrow$

$$2(\sin x + \sin 3x) = B_1 C_1 \sin nx$$

$$2[2 \sin 2x \cos x] = B_1 C_1 \sin nx \quad \left[\because \sin C + \sin D = 2 \sin \frac{C+D}{2} \sin \frac{C-D}{2}, \text{ if } C > D\right]$$

$$4 \sin 2x \cos x = B_1 C_1 \sin nx$$

Here $C = 3$, $D = 1$

On comparing, we have

$$B_1 C_1 = 4 \cos x, \quad n = 2$$

$$\boxed{\therefore u(x, t) = 4 \cos x \sin 2x \cos 2t}$$

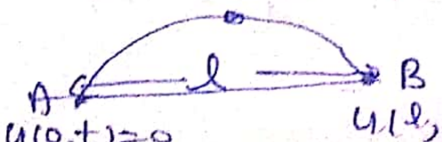
Required answer.

Q.4. A tightly stretched string with fixed end points $x = 0$, $x = l$ is initially in a position given by $u = u_0 \sin^3\left(\frac{\pi x}{l}\right)$. If it is released from rest from this position, Find the displacement $u(x, t) = ?$

Solution. The equation of the string is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——— (1)}$$

$u(x, 0) = ?$, $\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0$



The conditions are

$$u(0,t) = 0 \quad \text{--- (2)}$$

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0 \quad \text{--- (3)}$$

$$u(l,t) = 0 \quad \text{--- (4)}$$

$$u(x,0) = u_0 \sin^3\left(\frac{\pi x}{l}\right) \quad \text{--- (5)}$$

The solution of (1) is given by

$$u(x,t) = (A_1 \cos nx + B_1 \sin nx)(C_1 \cos nct + D_1 \sin nct) \quad \text{--- (6)}$$

$u(0,t) = 0 \Rightarrow$

$$0 = (A_1)(C_1 \cos nct + D_1 \sin nct) \quad \text{--- (7)}$$

$\Rightarrow A_1 = 0$ as $(C_1 \cos nct + D_1 \sin nct) \neq 0$

Equ (6) & (7) $\Rightarrow$

$$u(x,t) = \sin nx(B_1 C_1 \cos nct + B_1 D_1 \sin nct) \quad \text{--- (8)}$$

$$\frac{\partial u}{\partial t} = \sin nx[-B_1 C_1 \sin nct \cdot (nc) + nc \cdot B_1 D_1 \cos nct]$$

$\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0 \Rightarrow$

$0 = \sin nx[0 + nc \cdot B_1 D_1 \cos nct]$

$\Rightarrow B_1 D_1 = 0$ as $\sin nx \neq 0$ & $\cos nct \neq 0$

$\therefore$ Equ (8) $\Rightarrow$

$$u(x,t) = \sin nx[B_1 C_1 \cos nct] \quad \text{--- (9)}$$

Now, $u(l,t) = 0 \Rightarrow$

$0 = \sin nl \cdot B_1 C_1 \cos nct$

$\Rightarrow \sin nl = 0 = \sin m\pi \Rightarrow n = \frac{m\pi}{l}$ --- (10)

$\therefore$ (9) & (10) $\Rightarrow$

$$u(x,t) = \sin \frac{m\pi x}{l}\left[B_1 C_1 \cos\left(\frac{m\pi}{l} \cdot ct\right)\right] \quad \text{--- (11)}$$

or, $u_m(x,t) = \sin\frac{m\pi x}{l}\left[B_1 C_1 \cos\left(\frac{m\pi ct}{l}\right)\right]$ — (12)

Take B_1C_1 as B_m, we have

$$u_m(x,t) = b_m \sin\frac{m\pi x}{l}\cos\frac{m\pi ct}{l}$$

$$\Rightarrow \sum_{m=1}^{\infty} u_m(x,t) = \sum_{m=1}^{\infty} b_m \sin\frac{m\pi x}{l}\cos\frac{m\pi ct}{l} = u(x,t) \text{ say.}$$

i.e. $u(x,t) = \sum u_m(x,t) = \sum b_m \sin\frac{m\pi x}{l}\cos\left(\frac{m\pi ct}{l}\right)$ — (13)

Now it is given

$u(x,0) = u_0 \sin^3\left(\frac{\pi x}{l}\right)$

$\left[\begin{array}{l} \sin 3x = 3\sin x - 4\sin^3 x \\ \sin^3 x = \frac{3\sin x - \sin 3x}{4} \end{array}\right.$

$\Rightarrow u(x,0) = \frac{u_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]$ — (14)

Eqn (13) & (14) $\Rightarrow$ (put $t=0$ in 13)

$$\frac{u_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right] = \sum b_m \sin\frac{m\pi x}{l} \quad \left[\because \cos\frac{m\pi ct}{l} = 0 \text{ at } t=0\right]$$

$$\frac{u_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right] = b_1 \sin\frac{\pi x}{l} + b_2\sin\frac{2\pi x}{l} + b_3\sin\frac{3\pi x}{l} + b_4\sin\frac{4\pi x}{l} + \cdots$$

On comparing, we have

$b_1 = \frac{u_0}{4}\cdot 3$, $b_2 = 0$, $b_3 = -\frac{u_0}{4}$, $b_4 = 0 = b_5 = \cdots$ — (15)

$\therefore$ (13) & (15) $\Rightarrow$

$$\boxed{u(x,t) = \frac{3u_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l} - \frac{u_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l} = \frac{u_0}{4}\left[3\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l} - \sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l}\right]}$$

Required answer.

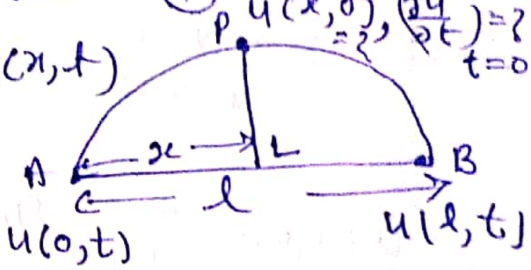
Q5. A string is stretched and fastened to two points l apart. Motion is started by displacing the string into the form $y = k(lx - x^2)$ from which it is released at time $t = 0$. Find the displacement of any point on the string at a distance of x from one end at time t.

Solution. The equation of the string is give by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left[\frac{\partial^2 u}{\partial x^2}\right] \quad \text{—— (1)}$$

Note: Here we suppose $y(x,t)$ by $u(x,t)$

the conditions are



$$u(0,t) = 0 \quad \text{—— (2)}$$

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0 \quad \text{—— (3)}$$

$$u(l,t) = 0 \quad \text{—— (4)}$$

$$u(x,0) = k(lx - x^2) \quad \text{—— (5)}$$

The solution of (1) is given by

$$u(x,t) = (A_1 \cos nx + B_1 \sin nx)(C_1 \cos nct + D_1 \sin nct) \quad \text{—— (6)}$$

put $x = 0$

$$u(0,t) = (A_1 + 0)(C_1 \cos nct + D_1 \sin nct)$$

$$0 = A_1(C_1 \cos nct + D_1 \sin nct)$$

$$\Rightarrow A_1 = 0 \quad \text{as } (C_1 \cos nct + D_1 \sin nct) \neq 0 \quad \text{—— (7)}$$

Equations (6) & (7) $\Rightarrow$

$$u(x,t) = \sin nx\,[B_1 C_1 \cos nct + B_1 D_1 \sin nct] \quad \text{—— (8)}$$

$$\frac{\partial u}{\partial t} = \sin nx\,[-nc\,B_1 C_1 \sin nct + nc\,B_1 D_1 \cos nct]$$

put $t=0$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sin nx\left[0 + nc\, B_1 D_1\right]$$

$$0 = \sin nx \cdot nc \cdot B_1 D_1 \Rightarrow B_1 D_1 = 0 \text{ as } \sin nx \neq 0$$

putting this value of $B_1 D_1$ in (8), we get

$$u(x,t) = \sin nx\left[B_1 C_1 \cos nct\right] \quad \text{—— (9)}$$

put $x = l$

$$u(l,t) = \sin nl\left[B_1 C_1 \cos nct\right]$$

$$0 = \sin nl\left[B_1 C_1 \cos nct\right]$$

$\Rightarrow \sin nl = 0$ as $\cos nct \neq 0$ & Take $B_1 C_1 \neq 0$

$$\Rightarrow \sin nl = 0 = \sin m\pi$$

$$\Rightarrow nl = m\pi \Rightarrow n = \frac{m\pi}{l} \quad \text{—— (10)}$$

, $m \in I$, Integers

$\therefore$ (9) and (10) $\Rightarrow$

$$u(x,t) = \sin\frac{m\pi x}{l} \cdot B_1 C_1 \cos\frac{m\pi ct}{l}$$

We can write

$$u(x,t) = \sum_{m=1}^{\infty} b_m \sin\frac{mx\pi}{l}\cos\frac{m\pi ct}{l} \quad \text{—— (11)}$$

, $b_m = B_1 C_1$, say

put $t=0$, We have

$u(x,0) = k(lx - x^2) \Rightarrow$

$$k(lx - x^2) = \sum_{m=1}^{\infty} b_m \sin\frac{m\pi x}{l} \quad \left[\because \cos\frac{m\pi ct}{l} = 0\right]$$

Clearly $\sum b_m \sin\frac{m\pi x}{l}$ is a fourier Sine Series

& $f(x) = k(lx - x^2)$

where

$$b_m = \frac{2}{l}\int_0^l f(x)\sin\frac{m\pi x}{l}\,dx$$

$$\Rightarrow b_m = \frac{2}{l}\int_0^l k(lx - x^2)\sin\frac{m\pi x}{l}\,dx \quad \left[\because f(x) = k(lx - x^2)\right]$$

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$$b_m = \frac{2k}{l}\left[-\frac{l}{m\pi}(lx-x^2)\cos\frac{m\pi x}{l}\Big|_0^l - \int_0^l (-)\frac{l}{m\pi}(l-2x)\cos\frac{m\pi x}{l}\,dx \right]$$

$$b_m = \frac{2k}{l}\left[0+0+\frac{l}{m\pi}\int_0^l (l-2x)\cos\frac{m\pi x}{l}\,dx \right]$$

$$= \frac{2k}{m\pi}\left[\frac{l}{m\pi}\cdot(l-2x)\sin\frac{m\pi x}{l}\Big|_0^l - \int_0^l \frac{l}{m\pi}(-2)\sin\frac{m\pi x}{l}\,dx \right]$$

$$= \frac{2k}{m\pi}\left[(-l\sin m\pi - l.0) + \frac{2l}{m\pi}\int_0^l \sin\frac{m\pi x}{l}\,dx \right]$$

$$= \frac{4kl}{m^2\pi^2}\int_0^l \sin\frac{m\pi x}{l}\,dx$$

$$= -\frac{4kl}{m^2\pi^2}\cdot\frac{l}{m\pi}\left[\cos\frac{m\pi x}{l}\right]_0^l$$

$$b_m = -\frac{4kl^2}{m^3\pi^3}\left[\cos m\pi - 1\right]$$

$$b_m = -\frac{4kl^2}{m^3\pi^3}\left[(-1)^m - 1\right]$$

$$b_m = \begin{cases} 0, & \text{if } m \text{ is even} \\ \dfrac{8kl^2}{m^3\pi^3}, & \text{if } m \text{ is odd.} \end{cases}$$

Putting This value of b_m into (11), we get

$$u(x,t) = \sum_{m=1}^{\infty} \frac{8kl^2}{m^3\pi^3}\sin\frac{m\pi x}{l}\cos\frac{m\pi ct}{l}$$

where m is odd natural number
i.e $m = 1, 3, 5, 7, 9, \dots$

which is required Fourier series.