CLASS: Msc MATHEMATICS IV SEM

SUBJECT CODE: H-4049
SUBJECT NAME: NUMBER THEORY

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① Number Theory
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Number Theoretic Function

(1) The function τ

$\tau : N \longrightarrow N$

such that $\tau(n) = \sum_{\substack{d|n \\ d\geq 1}} 1$, d is a divisor of n.

ex If $n = 15$, Find $\tau(15)$

divisors of 15 are 1, 3, 5, 15

Its mean's d's are 1, 3, 5, 15

i.e $d_1 = 1, d_2 = 3, d_3 = 5, d_4 = 15$

$\therefore \tau(15) = \sum_{d|15} 1 = 1+1+1+1$

[$\because d_1|15 \Rightarrow 1|15$
$d_2|15 \Rightarrow 3|15$
$d_3|15 \Rightarrow 5|15$
$d_4|15 \Rightarrow 15|15$

(2) If $n=18$

$\tau(18) = \sum_{\substack{d|18 \\ d\geq 1}} 1 = 1+1+1+1+1+1 = 6$

[$d_1=1, d_2=2$
$d_3=3, d_4=6$
$d_5=9, d_6=18$]

Note: ① $\tau(n)$ = Total no of divisors of n including 1 and n.

② $\tau(p_1^{\alpha_1})$ where p_1 is prime and $\alpha_1 \geq 0$

$= p_1^0 + p_1 + p_2^2 + \cdots + p_1^{\alpha_1}$

[$\because p_1^0 = 1 = d_1$ is a divisor of $p_1^{\alpha_1}$
$\Rightarrow d_2 = p_1$ is a divisor of $p_1^{\alpha_1}$
$d_3 = p_1^2$ is a divisor of $p_1^{\alpha_1}$
$d_4 = p_1^3$ is a divisor of $p_1^{\alpha_1}$
$\cdots$
$d_{\alpha_1+1} = p_1^{\alpha_1}$ is a divisor of $p_1^{\alpha_1}$

$\therefore$ Total no of divisors of $p_1^{\alpha_1}$ are $= (1 + p_1 + p_1^2 + p_1^3 + \cdots + p_1^{\alpha_1})$

$= \frac{(\alpha_1+1)}{1}$

$= (\alpha_1+1)$.

Similarly total no of divisors of p^{α_2} are $= (\alpha_2+1)$

total no of divisors of p^{α_k} are $= (\alpha_k+1)$

ii) if $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \dots \cdot p_k^{\alpha_k}$ then

$\tau(n) = (\alpha_1+1)(\alpha_2+1) \dots (\alpha_k+1)$.

Q.(1) Find $\tau(3655)$

Solution, we have

$3655 = 5 \times 17 \times 43$

$3655 = 5^{\alpha_1} \times 17^{\alpha_2} \times 43^{\alpha_3}$ where $\alpha_1 = 1 = \alpha_2 = \alpha_3$

$\therefore \tau(3655) = (\alpha_1+1)(\alpha_2+1)(\alpha_3+1)$

$= 2 \times 2 \times 2$

$= 8$

(Q2) Find $\tau(3000)$

Solu. we have $3000 = 2^3 \times 3^1 \times 5^3$

ii) $\tau(3000) = (3+1)(1+1)(3+1)$

$= 4 \times 2 \times 4$

$= 32$.

(Q3) Special Q

Verify that $\tau(n) = \tau(n+1) = \tau(n+2)$ holds for $n = 4503$.

Solution. we have

$\tau(n) = \tau(4503) = \tau(3 \times 19 \times 79)$ [$4503 = 3 \times 19 \times 79$]

$\tau(n) = (1+1)(1+1)(1+1) = 8$ — (1)

Now, $n+1 = 4504 = 2^3 \times 5^0 \times 53^1$

$\tau(n+1) = (3+1)(1+1) = 8$ — (2)

$n+2 = 4505 = 5^1 \times 17^1 \times 53^1$

$\tau(n+2) = \tau(4505) = 2 \times 2 \times 2 = 8$ — (3)

(1) (2) & (3) $\Rightarrow \tau(n) = \tau(n+1) = \tau(n+2)$ for $n = 4503$.

$\therefore$ verified.

Theorem 1. For any integer $n>1$
(2018) $\tau(n)$ is odd iff n is a perfect square

Proof. Let $n>1$ be an integer

"If part" Let n is a perfect square then we have

$$n = \left(p_1^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3} \cdots p_k^{a_k}\right)^2$$

$$\Rightarrow n = p_1^{2a_1} \cdot p_2^{2a_2} \cdot p_3^{2a_3} \cdots p_k^{2a_k}$$

$$\Rightarrow n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}, \text{ where each } \alpha_i \text{ is even.}$$

$$\therefore \tau(n) = (\alpha_1+1)(\alpha_2+1)\cdots(\alpha_k+1)$$

$$= (\text{even}+1)(\text{even}+1)\cdots(\text{even}+1)$$

$$= \text{An odd integer}$$

"Only if part" Suppose $\tau(n)$ is an odd integer

To prove n is a perfect square

Let $\tau(n) = (\alpha_1+1)(\alpha_2+1)\cdots(\alpha_k+1)$ ["∵ odd × odd = odd Always."]

where each (α_i+1) is odd

where $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, each p_i is prime ← (1)

each (α_i+1) is an odd integer

$\Rightarrow$ each α_i is an even integer

then (1) $\Rightarrow$

$$n = p_1^{2a_1} \cdot p_2^{2a_2} \cdots p_k^{2a_k}, \text{ where each } \alpha_i = 2a_i, \text{ say}$$

$$\Rightarrow n = \left(p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}\right)^2$$

$\Rightarrow n =$ A perfect square. proved.

The function σ

Let n be a positive integer greater than one

Then $\sigma(n)$ = Sum of all positive divisors of n.

i.e $\sigma(n) = \sum_{\substack{d|n \\ d\geq 1}} d = d_1 + d_2 + \cdots + d_k$ if each d_i divides n,

ex. (1) $\sigma(15) = 1+3+5+15 = 24$

where $d_1 = 1$, $d_2 = 3$, $d_3 = 5$, $d_4 = 15$

i.e 15 has only four divisors namely 1, 3, 5 and 15

Ex. (2)

$\sigma(18) = 1+2+3+6+9+18$

$= 39$

As 18 has 6 divisors namely $d_1 = 1$, $d_2 = 2$, $d_3 = 3$

$d_4 = 6$, $d_5 = 9$, $d_6 = 18$.

If n is a big positive integer i.e.

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \cdots p_k^{\alpha_k}$$

then

$\sigma(n)$ = sum of divisors of $\left[p_1^{\alpha_1} + p_2^{\alpha_2} + \cdots + p_k^{\alpha_k} \right]$

$$= \left(\frac{p_1^{\alpha_1+1} - 1}{p_1 - 1}\right)\left(\frac{p_2^{\alpha_2+1} - 1}{p_2 - 1}\right) \cdots \left(\frac{p_k^{\alpha_k+1} - 1}{p_k - 1}\right)$$

As each $p_i^{\alpha_i}$ has divisors

$1, p_i, p_i^2, p_i^3, \ldots, p_i^{\alpha_i}$ for $i = 1, 2, \ldots, k$

Sum of divisors of $p_1^{\alpha_1} = 1 + p_1 + p_1^2 + \cdots + p_1^{\alpha_1}$

Sum of divisors of $p_2^{\alpha_2} = 1 + p_2 + p_2^2 + \cdots + p_2^{\alpha_2}$

— — —

— — —

Sum of divisors of $p_k^{\alpha_k} = 1 + p_k + p_k^2 + \cdots + p_k^{\alpha_k}$

$$\therefore \sigma(n) = (1 + p_1 + \cdots + p_1^{\alpha_1})(1 + p_2 + p_2^2 + \cdots + p_2^{\alpha_2}) \cdots (1 + p_k + p_k^2 + \cdots + p_k^{\alpha_k})$$

$$\sigma(n) = \left[\frac{p_1^{\alpha_1+1} - 1}{p_1 - 1}\right]\left[\frac{p_2^{\alpha_2+1} - 1}{p_2 - 1}\right] \cdots \left[\frac{p_k^{\alpha_k+1} - 1}{p_k - 1}\right]$$

108
44
432
432

Q4. Find $\sigma(3655)$

Soln we have $3655 = 5^1 \times 17^1 \times 43^1$

$$\therefore \sigma(3655) = \left[\frac{5^2-1}{5-1}\right]\left[\frac{17^2-1}{17-1}\right]\left[\frac{43^2-1}{43-1}\right]$$

$$= (5+1)(17+1)(43+1) = 4752$$

Q5. Find $\sigma(3000)$.

Soln We have $3000 = 2^3 \times 3^1 \times 5^3$

$$\therefore \sigma(3000) = \left[\frac{2^4-1}{2-1}\right]\left[\frac{3^2-1}{3-1}\right]\left[\frac{5^4-1}{5-1}\right]$$

$$= 15 \times 4 \times 156 = 9360.$$

Q6. (Special Q.)
If $f(n) = n^2 + 2$ and $n = 6$
then show that $\sum_{d|6} f(d) = \sum_{d|6} f\left(\frac{6}{d}\right)$

Solution. Given $f(n) = n^2 + 2$ ——— (1)

Let $n = 6$

divisors of 6 are $d_1 = 1, d_2 = 2, d_3 = 3, d_4 = 6$

$$\therefore \sum_{d|6} f(d) = f(d_1) + f(d_2) + f(d_3) + f(d_4)$$

$$= d_1^2 + 2 + d_2^2 + 2 + d_3^2 + 2 + d_4^2 + 2$$

[$\because f(n) = n^2 + 2$ $\therefore f(d) = d^2 + 2$ etc]

$$= (1^2+2) + (2^2+2) + (3^2+2) + (6^2+2)$$

~~$= 3 \times 6 \times 11 \times 38$~~

$$= 3 + 6 + 11 + 38 = 58 \quad \text{——— (2)}$$

$$\sum_{d|6} f(d) = f\left(\frac{6}{d_1}\right) + f\left(\frac{6}{d_2}\right) + f\left(\frac{6}{d_3}\right) + f\left(\frac{6}{d_4}\right)$$

$$\sum_{d|6} f\left(\frac{6}{d}\right) = f\left(\frac{6}{1}\right) + f\left(\frac{6}{2}\right) + f\left(\frac{6}{3}\right) + f\left(\frac{6}{6}\right)$$

$$= f(6) + f(3) + f(2) + f(1)$$

$$= f(1) + f(2) + f(3) + f(6)$$

$$\sum_{d|6} f\left[\frac{6}{d}\right] = (1^2+2) + (2^2+2) + (3^2+2) + (6^2+2) = 58 \quad \text{——— (3)}$$

From (2) and (3), we have

$$\sum_{d|6} f(d) = \sum_{d|6} f\left(\frac{6}{d}\right) \quad \text{if } f(n) = n^2 + 2 \text{ and } n = 6.$$

Theorem. For any integer $n > 1$,

$\sigma(n)$ is odd iff n is a perfect square or twice of a perfect square

Proof. For any integer $n > 1$

We have

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \cdots \cdot p_k^{\alpha_k}, \text{ each } p_i \text{ is prime}, \alpha_i \geq 0$$

$$\Rightarrow \sigma(n) = \left[\frac{p_1^{\alpha_1+1} - 1}{p_1 - 1}\right]\left[\frac{p_2^{\alpha_2+1} - 1}{p_2 - 1}\right] \cdots \left[\frac{p_k^{\alpha_k+1} - 1}{p_k - 1}\right]$$

$$\sigma(n) = \left[1 + p_1 + p_1^2 + \cdots + p_1^{\alpha_1}\right]\left[1 + p_2 + p_2^2 + \cdots + p_2^{\alpha_2}\right] \cdots \left[1 + p_k + p_k^2 + \cdots + p_k^{\alpha_k}\right]$$

Now,

$\sigma(n)$ is odd $\Leftrightarrow \left[1 + p_1 + p_1^2 + \cdots + p_1^{\alpha_1}\right]\left[1 + p_2 + p_2^2 + \cdots + p_2^{\alpha_2}\right] \cdots \left[1 + p_k + p_k^2 + \cdots + p_k^{\alpha_k}\right]$

$\sigma(n)$ is odd $\Leftrightarrow$ A product of odd integers

$\sigma(n)$ is odd $\Leftrightarrow$ Each $\left[1 + p_i + p_i^2 + \cdots + p_i^{\alpha_i}\right]$ is odd, $i = 1, 2, \cdots k$.

$\sigma(n)$ is odd $\Leftrightarrow$ Each α_i is even

$\sigma(n)$ is odd $\Leftrightarrow \alpha_i = 2m_i$ [α_i is even means 2 must divide α_i]

$\sigma(n)$ is odd $\Leftrightarrow n = p_1^{2m_1} \cdot p_2^{2m_2} \cdots p_k^{2m_k}$

$\sigma(n)$ is odd $\Leftrightarrow n = \left(p_1^{m_1} \cdot p_2^{m_2} \cdots p_k^{m_k}\right)^2$ — (1)

$\sigma(n)$ is odd $\Leftrightarrow n$ is a perfect square

if $p_1 = 2$ then (1) $\Rightarrow$

$\sigma(n)$ is odd $\Leftrightarrow n = 2^{2m_1} \cdot p_2^{2m_2} \cdot p_3^{2m_3} \cdots p_k^{2m_k}$ — (2)

if $2m_1$ is odd let $2m_1 = \alpha_1 = 2m + 1$

then (2) $\Rightarrow$

$\sigma(n)$ is odd $\iff n = 2^{2m+1}\left(p_2^{m_2} \cdot p_3^{m_3} \cdots p_k^{m_k}\right)^2$

$\iff n = 2 \cdot \left(2^{m} \cdot p_2^{m_2} \cdot p_3^{m_3} \cdots p_k^{m_k}\right)^2$

$\iff n$ is a twice of perfect square.

Theorem. 1. Prove that τ and σ are multiplicative

Proof. We know that an arithmatic function f is said to be multiplicative if $f(mn) = f(m) f(n)$ where $(m,n) = 1$.

Let $m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \cdots p_s^{\alpha_s}$, $\alpha_s \geq 0$, each p_i is prime

$n = q_1^{\beta_1} \cdot q_2^{\beta_2} \cdot q_3^{\beta_3} \cdots q_s^{\beta_s}$

where $\beta_i \geq 0$ and each q_j is prime

For $m = n = 1$, the result is obvious

Now

$$mn = \left(p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \cdots p_s^{\alpha_s}\right)\left(q_1^{\beta_1} \cdot q_2^{\beta_2} \cdot q_3^{\beta_3} \cdots q_s^{\beta_s}\right)$$

$$\Rightarrow \tau(mn) = [(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_s+1)][(\beta_1+1)(\beta_2+1)\cdots(\beta_s+1)]$$

$$\tau(m,n) = \tau(m) \cdot \tau(n) \quad \text{---------- (1) as } (m,n) = 1$$

Also

$$\sigma(mn) = \left[\frac{p_1^{\alpha_1+1}-1}{p_1-1} \cdot \frac{p_2^{\alpha_2+1}-1}{p_2-1} \cdots \frac{p_s^{\alpha_s+1}-1}{p_s-1}\right]\left[\frac{q_1^{\beta_1+1}-1}{q_1-1} \cdot \frac{q_2^{\beta_2+1}-1}{q_2-1} \cdots \frac{q_s^{\beta_s+1}-1}{q_s-1}\right]$$

$$\sigma(m,n) = \sigma(m) \cdot \sigma(n) \quad \text{---------- (2)} \quad \text{as } (m,n) = 1$$

i) equations (1) and (2) $\Rightarrow$

τ and σ are multiplicative functions.

Note. Let $d = mn$ — (1)
$d_1 = m$ — (2)
$d_2 = n$ — (3)

(1), (2) and (3) $\Rightarrow$ $d = d_1 d_2$ & $(d_1, d_2) = 1$

$\Rightarrow d_1 \mid m$, $d_2 \mid n \Rightarrow d_1 d_2 \mid mn$.

Theorem 03. If f is a multiplicative function and

$$S(n) = \sum_{d|n} f(d)$$

then $S(n)$ is also multiplicative.

Proof. Let f is multiplicative function

Let $S(n) = \sum_{d|n} f(d)$

Let m and n be any two prime integers such that $(m,n) = 1$

then $S(mn) = \sum_{\substack{d_1|m \\ d_2|n}} f(d_1 d_2)$

$$S(mn) = \sum_{d_1|m} f(d_1) \cdot \sum_{d_2|n} f(d_2)$$

$$S(mn) = S(m) \cdot S(n)$$

$\Rightarrow S$ is also multiplicative function.

Lattice point. If is the point whose all co-ordinates are integers.

Example. Show that $\tau(n)$ is equal to the lattice points on the hyperbola $xy = n$ lying in the first quadrant.

Solution. Suppose $d_1, d_2, \cdots d_r$ be the divisors of n

Given equation of hyperbola

$$xy = n \qquad \text{---------- (1)}$$

clearly $\left(d_1, \frac{n}{d_1}\right), \left(d_2, \frac{n}{d_2}\right), \cdots \left(d_r, \frac{n}{d_r}\right)$

Called lattice points satisfy (1).

$\therefore \tau(n) =$ no of lattice points on $xy = n$.

Behaviour of $\tau(n)$ as $n \to \infty$

We know if p is prime then

$$\tau(p) = 2$$

if p is not prime say

$$p = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_r^{\alpha_r}, \quad \alpha_i \geq 0, \text{ each } p_i \text{ is prime}$$

$$\tau(p) = (\alpha_1+1)(\alpha_2+1)\cdots(\alpha_r+1)$$

$$\Rightarrow \tau(p) \geq 2$$

Here we observe that if $n = p^a$

then $\tau(n) = (a+1)$

if $a \to \infty$ $\tau(n) \to \infty$

i.e $\lim_{n \to \infty} \tau(n) = \infty$ [because a will depend on n]

The behaviour of $\tau(n)$ is irregular.

Behaviour of $\sigma(n)$ as $n \to \infty$

We know $\sigma(n)$ = sum of divisors of n

$$\Rightarrow \sigma(n) > n$$

as $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$

$$\sigma(n) = \left(\frac{p_1^{\alpha_1+1}-1}{p_1-1}\right)\left(\frac{p_2^{\alpha_2+1}-1}{p_2-1}\right)\cdots\left(\frac{p_r^{\alpha_r+1}-1}{p_r-1}\right)$$

ex $n = 10 \Rightarrow 10 = 2 \times 5$

$$\sigma(10) = \frac{2^2-1}{2-1} \cdot \frac{5^2-1}{5-1} = 3 \times 6 = 18$$

$$\Rightarrow \sigma(10) > 10$$

Hence $\lim_{n \to \infty} \sigma(n) = \infty$.

The behaviour of $\sigma(n)$ is regular.

The Mobious function (or Mobious Inversion formula)

$\mu : N \longrightarrow \{0, 1, -1\}$

such that

$\mu(1) = 1$

$\mu(n) = 0$, if n is divisible by p^2, p is prime

$\mu(n) = (-1)^k$ if $n = p_1 \cdot p_2 \cdot \cdots p_k$, each p_i is prime, $1 \le i \le k$, k is no of distinct primes

$\mu(2) = (-1)^1$ [$\because n = 2 = p_1$, p_1 is only one prime]
$= -1$

$\mu(3) = (-1)^1$
$= -1$

$\mu(4) = 0$ as $2^2 | 4$ [$\because p^2 | 4$ i.e 4 is divisible by the square of a prime no]

$\mu(5) = -1$

$\mu(6) = (-1)^2$ [$\because 6 = 2 \times 3 = p_1 \times p_2$, no of primes = 02]

$\mu(7) = -1$

$\mu(8) = 0$ [$\because$ 8 is divisible by 2^2 i.e (prime no)^2, 2 is prime]

$\mu(9) = 0$ [$3^2 | 9$, 3 is prime]

$\mu(10) = (-1)^2$ [$10 = 2 \times 5 = p_1 \times p_2$ no of distinct primes = 02]

$\mu(10) = 1$

$\mu(11) = -1$ [$11 = p_1$, no of prime = 0]

$\mu(12) = 0$ [$\because$ 12 is divisible by (prime)^2 i.e 12 is divisible by $(2)^2$, 2 is prime]

$\mu(13) = -1$

and so on.

Theorem 1. Prove Mobious fanction is multiplicative
i.e $\mu(mn) = \mu(m) \cdot \mu(n)$ if $(m,n) = 1$

Proof. Let m, n be any two positive integers such that
$(m,n) = 1$
Then three cases arise
(1) $m = 1$ (2) m is divisible by p^2, p is prime
(3) $m = p_1 p_2 \cdots p_k$ each p_i and q_i are different prime
$n = q_1 q_2 \cdots q_k$, $p_i \neq q_j$, $1 \leq i = j \leq k$

Case (1) $m = 1$ then

$$\begin{aligned} \mu(mn) &= \mu(1 \cdot n) && [1 \cdot n = n] \\ &= \mu(n) && [1 \cdot \mu(n) = \mu(n)] \\ &= 1 \cdot \mu(n) && [\because \mu(1) = 1, m = 1] \\ &= \mu(m)\mu(n) \end{aligned}$$

Case (2)
(2) if $p^2 \mid m$ then $\mu(m) = 0$
i) $\mu(m) \cdot \mu(n) = 0 \cdot \mu(n)$
$= 0$
and $\mu(mn) = 0$ [as $p^2 \mid m \Rightarrow p^2 \mid mn$]
ii) $\mu(mn) = \mu(m) \cdot \mu(n)$

Case (3) if $m = p_1 p_2 \cdots p_k$, p_i, q_i are all different primes
& $n = q_1 q_2 \cdots q_k$

$$\mu(mn) = \mu(p_1 p_2 \cdots p_k \cdot q_1 q_2 \cdots q_k)$$

$$\mu(mn) = (-1)^k \cdot (-1)^k$$
$$\mu(mn) = (-1)^{2k} \quad \text{---------- (1)}$$
$$\mu(mn) = 1$$

$$\mu(m) = \mu(p_1 p_2 \cdots p_k) = (-1)^k$$
$$\mu(n) = \mu(q_1 q_2 \cdots q_k) = (-1)^k$$
$$\mu(m)\mu(n) = (-1)^k (-1)^k = (-1)^{2k} = 1 \quad \text{---------- (2)}$$

From (1) & (2)
$$\mu(mn) = \mu(m)\mu(n).$$

Theorem. F. Morten's Lemma For each positive integer $n \geq 1$

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } n>1 \end{cases}$$

Not for exam to write केवल Th. को समझने के लिए

For $n>1$ $\sum_{d|n} \mu(d) = 0$ का अर्थ है कि n के divisors के अलग-अलग μ function निकाले जाए और यदि उनका सम (जोड़) किया जाये तो 0 आता है

जैसे $n=6$ इसके divisors $d_1=1, d_2=2, d_3=3, d_4=6$

तो $\mu(d_1)+\mu(d_2)+\mu(d_3)+\mu(d_4)=0$

i.e. $\mu(1)+\mu(2)+\mu(3)+\mu(6) = 1-1-1+1=0$

$\because \mu(1)=1, \mu(2)=-1, \mu(3)=-1$
$\mu(6)=1$

Proof. For $n=1$

$$\sum_{d|n} \mu(d) = \sum_{d|1} \mu(d) = \mu(1) = 1 \quad [\because 1 \text{ is only divisor of } 1]$$

For $n>1$, we can write

$n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, each p_i is prime $1 \leq i \leq r$, $k_i \geq 0$,

$$\sum_{d|n} \mu(d) = \mu(1) + \sum_{i=1} \mu(p_i) + \sum_{(i \neq j)} \mu(p_i p_j) + \cdots + \sum \mu(p_1 p_2 \cdots p_r) + \text{terms have zero } \mu \text{ function}$$

[$\because$ possible di's of n are
$1, p_i, p_i p_j, p_i p_j p_k, \ldots p_1 p_2 \cdots p_r$ and remaining divisors divisible by p^2, p is prime having μ function as zero]

$$\sum_{d|n} \mu(d) = 1 + r(-1) + \sum_{2}^{r} (-1)^2 + \cdots + (-1)^r + 0 + 0 + \cdots 0$$

$$\sum_{d|n} \mu(d) = (1+x)^r, \quad x=-1$$

$$\sum_{d|n} \mu(d) = 0$$

, proved.

Theorem 03 Mobius inversion formula

Let $F(n) = \sum_{d|n} f(d)$

then $f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right)$

OR $f(n) = \sum_{d|n} \mu\left(\frac{d}{n}\right) F(n)$

Proof. Let $F(n) = \sum_{d|n} f(d)$ ——— (1)

To prove $f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right)$

$$RHS = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right)$$

$$= \sum_{d|n} \mu(d) \sum_{c|\frac{n}{d}} f(c) \quad [\text{using concept of (1)}]$$

$$= \sum_{d|n} \sum_{c|\frac{n}{d}} \mu(d) f(c)$$

$$= \sum_{c|n} f(c) \sum_{d|\frac{n}{c}} \mu(d) \quad \left[\begin{array}{l} c|\frac{n}{d} \Rightarrow \frac{n}{d} = ck, k \in I \\ \Rightarrow \frac{n}{c} = dk \\ \Rightarrow d|\frac{n}{c} \end{array}\right]$$

put $c = n$

$$= f(n) \cdot \sum_{d|1} \mu(d)$$

$$= f(n) \cdot 1 \quad [\because d = 1 \text{ is only factor of } 1]$$

$$= LHS$$

Similarly $f(n) = \sum_{d|n} \mu\left(\frac{d}{n}\right) F(n)$.

Theorem.05 If $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ then $\sum_{d|n} |\mu(d)| = 2^k$

Soln Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$

For $n > 1$

$$\sum_{d|n} |\mu(d)| = |\mu(1)| + \sum_{i=1}^{k} |\mu(p_i)| + \sum_{\substack{i<j \\ i\neq j \\ i=1}}^{k} |\mu(p_i p_j)|$$

$$+ \cdots + |\mu(p_1 \cdot p_2 \cdots p_k)| + \text{ } \mu \text{ function of } n \text{ which is divisible by } p_i^2.$$

$$\sum_{d|n} \mu(d) = 1 + |{}^{k}C_1(-1)| + |{}^{k}C_2(-1)^2| + |{}^{k}C_3(-1)^3| + \cdots + |(-1)^k| + 0 + 0 + \cdots$$

$$= (1+x)^k \text{ where } x = +1$$

$$= 2^k.$$

Theorem 6: If $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ then

prove $\sum_{d|n} \mu(d) \cdot \frac{1}{d} = \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_n}\right)$

(1) if $n = p_1^{\alpha_1}$

Its divisors are $1, p_1, p_1^2, p_1^3, \cdots p_1^{\alpha_1}$

i) $$\sum_{d|n} \mu(d) \frac{1}{d} = \mu(1) \cdot \frac{1}{1} + \frac{\mu(p_1)}{p_1} + \frac{\mu(p_1^2)}{p_1^2} + \cdots \frac{\mu(p_1^{\alpha_1})}{p_1^{\alpha_1}}$$

$$= 1 - \frac{1}{p_1} + 0 + 0 + \cdots,$$

$$= \left(1 - \frac{1}{p_1}\right)$$

(2) If $n = p_2^{\alpha_2}$

$$\sum_{d|n} \mu(d) \cdot \frac{1}{d} = \left(1 - \frac{1}{p_2}\right)$$

If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$

ii) $$\sum_{d|n} \mu(d) \frac{1}{d} = \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)$$

Exp (1) If $n = 2\times3\times5$

then $\sum_{d|n} u(d)\cdot\frac{1}{d} = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right) = \frac{1}{2}\times\frac{2}{3}\times\frac{4}{5} = \frac{4}{15}$

$$LHS = \sum_{d|n} u(d)\cdot\frac{1}{d} = \sum_{d|30} u(d)\cdot\frac{1}{d}$$

$$= \frac{u(1)}{1} + \frac{u(2)}{2} + \frac{u(3)}{3} + \frac{u(5)}{5} + \frac{u(6)}{6} + \frac{u(10)}{10} + \frac{u(15)}{15} + \frac{u(30)}{30}$$

$$= 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{u(5)\,u(6)}{30}$$

$$= 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} - \frac{1}{30}$$

$$= \frac{30-15-10-6+5+3+2-1}{30} = \frac{60-32}{30} = \frac{8}{30} = \frac{4}{15}$$

$\therefore$ LHS = RHS.

EX.02. For each positive integer n, show

$$u(n)\,u(n+1)\,u(n+2)\,u(n+3) = 0$$

Sol. To prove

$$u(n)\,u(n+1)\,u(n+2)\,u(n+3) = 0$$

$$LHS = u(n)\,u(n+1)\,u(n+2)\,u(n+3)$$

$$= u[n(n+1)(n+2)(n+3)] \quad [\because u \text{ is multiplicative}]$$

$$= 0 \quad \text{as} \quad 2^2 \mid \{n(n+1)(n+2)(n+3)\}$$

$$= RHS$$

$\therefore u(n)\,u(n+1)\,u(n+2)\,u(n+3) = 0.$

Ex.03 Find a positive integer n such that

$$u(n)+u(n+1)+u(n+2)=3$$

Sol Let $n=33$ (or 85 or 93)

$$u(n)=u(33)=u(3\times 11)=(-1)^2=1$$

$$u(n+1)=u(34)=u(2\times 17)=(-1)^2=1$$

$$u(n+2)=u(35)=u(5\times 7)=(-1)^2=1$$

$$\Rightarrow u(n)+u(n+1)+u(n+2)=3$$

Similarly we can prove for $n=85$ and $n=93$.

Ex.04 If n is positive integer such that $n\geq 3$

Then $\sum_{k=1}^{n} u(\lfloor k \rfloor)=1$

Soln To prove

$$u(\lfloor 1\rfloor)+u(\lfloor 2\rfloor)+\cdots+u(\lfloor n\rfloor)=1$$

$$LHS=u(\lfloor 1\rfloor)+u(\lfloor 2\rfloor)+u(\lfloor 3\rfloor)+u(\lfloor 4\rfloor)+\cdots+u(\lfloor n\rfloor)$$

$$=u(1)+u(2)+u(6)+\text{terms whose } u \text{ function is zero as } 2^2\mid \lfloor n+1\rfloor,\ n\geq 3$$

$$=1-1+1$$

$$=1$$

$$=RHS,\ \text{proved.}$$

Note $u(\lfloor 4\rfloor)=u(24)=0$ as $2^2\mid 24$

$u(\lfloor 5\rfloor)=u(5\times 24)=0$ as $2^2\mid 25$

$\vdots$

$u(\lfloor n\rfloor)=0$ as $n\geq 3$.

Greatest integer function. $[x]$

The greatest integer function $[x]$ is the unique integer less than or equal to x.

ex if $n \le x \le n+1$ where x is any fraction or integer
and n is integer $\le x$
$(n+1)$ is the integer $> x$.

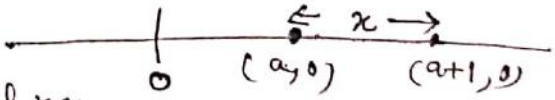
then $[x] = x$ if x is not fraction
$[x] = n$ if x is a fraction

ex ① $x = \frac{1}{3}$
then $0 < \frac{1}{3} < 1 \Rightarrow [\frac{1}{3}] = 0$

② $x = 2$ then $[x] = 2$

③ $x = \frac{20}{3} \Rightarrow 6 < \frac{20}{3} < 7 \Rightarrow [\frac{20}{3}] = 6$ etc.

Note. $x = [x] + \theta$, $0 \le \theta < 1$. (As $[x]$ is greatest integer less than or equal to x)



Theorem. For any x, x is real no. $[x] = a$
$[x] \le x$ such that x is greatest integer

Prove ① $[x+n] = [x] + n$, $n \in I$

② $[x] + [-x] = \begin{cases} 0, & \text{if } x \in I \\ -1, & \text{otherwise} \end{cases}$

Proof. ① We know that $[x] + \theta = x$, $0 \le \theta < 1$

$\Rightarrow [x] + \theta + n = x + n$, $n \in N$

$\Rightarrow [x+n] = [x] + n$ [$\because 0 \le \theta < 1$]

② To prove $[x] + [-x] = 0$ if $x \in I$

We have $[x] = x$ if $x \in I$

$[-x] = -x$ if $x \notin I$

$\Rightarrow [x] + [-x] = 0$, if $x \in I$

if $n \leq x \leq n+1 \Rightarrow [x] = n$ — (1)

$\Rightarrow -n \geq -x \geq -(n+1)$

$\Rightarrow -(n+1) \leq -x \leq -n \Rightarrow [-x] = -(n+1)$ — (2)

(1) & (2) $\Rightarrow$

$[x] + [-x] = n - n - 1 = -1$ if $x \notin I$

Theorem. 04. The highest power of a prime contained in $\lfloor n$ is $h_1 + h_2 + \cdots + h_k$ where $h_1 = \left[\frac{n}{p}\right]$, $h_2 = \left[\frac{h_1}{p}\right]$, $\cdots$, $h_k = \left[\frac{h_{k-1}}{p}\right]$

Proof. The highest powers of p in $\lfloor n$ is

$$\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots + \left[\frac{n}{p^k}\right]$$

Now $\left[\frac{n}{p}\right] = h_1$, say

then $\left[\frac{n}{p^2}\right] = \left[\frac{1}{p} \cdot \left[\frac{n}{p}\right]\right] = \left[\frac{1}{p} \cdot h_1\right] = \left[\frac{h_1}{p}\right] = h_2$, say

$$\left[\frac{n}{p^3}\right] = \left[\frac{1}{p} \cdot \left[\frac{n}{p^2}\right]\right] = \left[\frac{h_2}{p}\right] = h_3$$

$$\vdots$$

$$\left[\frac{n}{p^k}\right] = \left[\frac{1}{p}\left[\frac{n}{p^{k-1}}\right]\right] = \left[\frac{h_{k-1}}{p}\right]$$

Therefore, highest power of a prime contained in $\lfloor n$ is $h_1 + h_2 + \cdots + h_k$.

Ex.01. Find the highest power of 3 contained in $\lfloor 40$.

Solution. Required answer is

$$= \left[\frac{40}{3}\right] + \left[\frac{40}{3^2}\right] + \left[\frac{40}{3^3}\right] + \left[\frac{40}{3^4}\right] + \cdots$$

$$= 13 + \left[\frac{13}{3}\right] + \left[\frac{40}{3^3}\right] + 0$$

$$= 13 + 4 + \left[\frac{4}{3}\right] + 0$$

$$= 13 + 4 + 1$$

$$= 18$$

Ex-02 Find the highest power of 7 contained in $\lfloor 2000$.

Sol: Let $H = h_1 + h_2 + h_3$ be the highest power of 7 in $\lfloor 2000$

Let $h_1 = \left[\frac{2000}{7}\right] = 285$

$h_2 = \left[\frac{285}{7}\right] = 40$

$h_3 = \left[\frac{40}{7}\right] = 5$

$\therefore$ Required answer $= 285 + 40 + 5$

$= 330.$

Ex03 Find the highest power of 13 contained in $\lfloor 20,000$.

Sol: Req. highest power of 13 in $\lfloor 20,000$ is

$H = h_1 + h_2 + h_3 + h_4 + \cdots$

where $h_1 = \left[\frac{20000}{13}\right]$, $h_2 = \left[\frac{20,000}{13^2}\right]$, $h_3 = \left[\frac{20000}{13^3}\right]$, ...

$h_1 = \left[\frac{20000}{13}\right] = 1538$

$h_2 = \frac{1538}{13} = 118$

$h_3 = \frac{118}{13} = 9$

$h_4 = \left[\frac{9}{13}\right] = 0$

$\therefore \quad H = 1538 + 118 + 9$

$H = 1665$

Ex Find highest power of 12 in $\lfloor 500$

Sol $12 = 2^2 \times 3$

highest power of 2 in $\lfloor 500 = \left[\frac{500}{2}\right] + \left[\frac{500}{2^2}\right] + \left[\frac{500}{2^3}\right] + \cdots$

$H_1 = 250 + \left[\frac{125}{2}\right] + \left[\frac{62}{2}\right] + \left[\frac{31}{2}\right] + \left[\frac{15}{2}\right] + \left[\frac{7}{2}\right] + \left[\frac{3}{2}\right]$

$H_1 = 250 + 125 + 31 + 15 + 7 + 3 + 1 = 494$

Similarly highest powers of 3 in $\lfloor 500$

$H_2 = \left[\frac{500}{3}\right] + \left[\frac{500}{3^2}\right] + \left[\frac{500}{3^3}\right] + \left[\frac{500}{3^4}\right] + \cdots$

$= 166 + \left[\frac{166}{3}\right] + \left[\frac{55}{3}\right] + \left[\frac{18}{3}\right] + \left[\frac{6}{3}\right] + 0$

$= 166 + 55 + 18 + 6 + 2 + 0$

$= 247.$

$\lfloor 500 = 2^{494} \times 3^{247} \times t$

$= (2^2)^{247} \times 3^{247} \times t = (12)^{247} \times t$

$\therefore$ highest powers of 12 in $\lfloor 500 = 247.$

Note. Find highest powers of 18 in $\lfloor 500$

$$\lfloor 500 = 2^{494} \times 3^{247} \times t$$

$$= 2^{125} \times 2^{371} \times (3^2)^{123} \times 3 \times t$$

$$= (2+3^2)^{123} \times 2^{371} \times 3 \times t$$

$$= (18)^{123} \times 2^{371} \times 3 \times t$$

$\therefore$ Highest power of 18 in $\lfloor 500$ is 123.