SUBJECT: MATHEMATICS

PAPER: ADVANCED MATHEMATICAL METHODS
PAPER CODE:H 3051(PAPER III)
CLASS: MSc III SEMESTER
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① Show $\phi(x) = x e^x$ is a solution of

$$\phi(x) = \sin x + 2\int_0^x \cos(x-\xi)\phi(\xi)d\xi$$

Soln. Given

$$\phi(x) = \sin x + 2\int_0^x \cos(x-\xi)\phi(\xi)d\xi \quad \text{--- (1)}$$

To show $\phi(x) = xe^x$ is a solution of (1).

$$RHS = \sin x + 2\int_0^x \cos(x-\xi)\phi(\xi)d\xi \quad [\because \cos(A-B) = \cos A\cos B + \sin A\sin B]$$

$$RHS = \sin x + 2\int_0^x \cos x\cos\xi\,\phi(\xi)d\xi + 2\int_0^x \sin x\sin\xi\,\phi(\xi)d\xi$$

$$RHS = \sin x + 2\cos x\int_0^x \cos\xi\cdot\xi e^{\xi}d\xi + 2\sin x\int_0^x \sin\xi\cdot\xi e^{\xi}d\xi$$

$$RHS = \sin x + 2\cos x\int_0^x \xi\{e^{\xi}\cos\xi\}d\xi + 2\sin x\int_0^x \xi\{e^{\xi}\sin\xi\}d\xi$$

$$RHS = \sin x + 2\cos x\left[\xi\cdot\frac{e^{\xi}}{2}(\cos\xi+\sin\xi)\Big|_0^x - \int_0^x \frac{e^{\xi}}{2}(\cos\xi+\sin\xi)d\xi\right]$$
$$+ 2\sin x\left[\xi\cdot\frac{e^{\xi}}{2}(\sin\xi-\cos\xi)\Big|_0^x - \int_0^x \frac{e^{\xi}}{2}(\sin\xi-\cos\xi)d\xi\right]$$

$$RHS = \sin x + 2\cos x\left[\frac{xe^x}{2}(\cos x+\sin x) - 0 - \frac{1}{2}\int_0^x e^{\xi}(\cos\xi+\sin\xi)d\xi\right]$$
$$+2\sin x\left[\frac{xe^x}{2}(\sin x-\cos x) - 0 - \frac{1}{2}\int_0^x e^{\xi}(\sin\xi-\cos\xi)d\xi\right]$$

$$RHS = \sin x + xe^x(\cos^2 x + \cancel{\sin x\cos x} + \sin^2 x - \cancel{\sin x\cos x})$$
$$-\cos x\int_0^x e^{\xi}(\cos\xi+\sin\xi)d\xi - \sin x\int_0^x e^{\xi}(\sin\xi-\cos\xi)d\xi$$

$$RHS = \sin x + xe^x - (\cos x+\sin x)\int_0^x e^{\xi}\sin\xi\,d\xi + (\sin x-\cos x)\int_0^x e^{\xi}\cos\xi\,d\xi$$

$$RHS = \sin x + xe^x - (\cos x+\sin x)\left\{\frac{e^{\xi}}{2}(\sin\xi-\cos\xi)\right\}_0^x + (\sin x-\cos x)\left[\frac{e^{\xi}}{2}(\cos\xi+\sin\xi)\right]_0^x$$

$$RHS = \sin x + x e^x - \left(\frac{\cos x + \sin x}{2}\right)\left[e^x(\sin x - \cos x) - (0-1)\right]$$
$$+ \left(\frac{\sin x - \cos x}{2}\right)\left[e^x(\cos x + \sin x) - (1+0)\right]$$

$$= \sin x + x e^x - \cancel{\frac{e^x}{2}(\sin^2 x - \cos^2 x)} - \frac{1}{2}(\cos x + \sin x)$$
$$+ \cancel{\frac{e^x}{2}(\sin^2 x - \cos^2 x)} - \frac{1}{2}(\sin x - \cos x)$$

$$= \sin x + x e^x - \cancel{\frac{1}{2}\cos x} - \frac{1}{2}\sin x - \frac{1}{2}\sin x + \cancel{\frac{1}{2}\cos x}$$

$$RHS = \sin x + x e^x - \sin x$$
$$= x e^x = \phi(x)$$
$$= LHS.$$

which shows that $\phi(x)$ is a solution of ①.

[Formula used. $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$

$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$]

② Show $\phi(x) = (1+x^2)^{-3/2}$ is a solution of

$$\phi(x) = \frac{1}{1+x^2} - \int_0^x \frac{\xi}{1+x^2}\phi(\xi)\,d\xi$$

Solution. Given $\phi(x) = \frac{1}{1+x^2} - \int_0^x \frac{\xi}{1+x^2}\cdot\phi(\xi)\,d\xi$ — ①

To show $\phi(x) = (1+x^2)^{-3/2}$ is a solution of ①

$$RHS = \frac{1}{1+x^2} - \int_0^x \frac{\xi}{1+x^2}\phi(\xi)\,d\xi$$

$$RHS = \frac{1}{1+x^2} - \frac{1}{1+x^2}\int_0^x \xi\cdot\frac{1}{(1+\xi^2)^{3/2}}\,d\xi$$

put $1+\xi^2 = t^2$

$\xi\,d\xi = t\,dt$

$$RHS = \frac{1}{1+x^2} - \frac{1}{1+x^2}\int \frac{t\,dt}{(t^2)^{3/2}} \quad , t = \sqrt{1+\xi^2}$$

$$RHS = \frac{1}{1+x^2} - \frac{1}{1+x^2}\int \frac{1}{t^2}\,dt \quad , t = \sqrt{1+\xi^2}$$

$$RHS = \frac{1}{1+x^2} - \frac{1}{1+x^2}\left[-\frac{1}{t}\right]_0^x \quad , t = \sqrt{1+\xi^2}$$

$$RHS = \frac{1}{1+x^2} + \frac{1}{1+x^2}\left[\frac{1}{\sqrt{1+\xi^2}}\right]_0^x$$

$$RHS = \frac{1}{1+x^2} + \frac{1}{1+x^2}\left[\frac{1}{\sqrt{1+x^2}} - \frac{1}{\sqrt{1+0}}\right]$$

$$RHS = \frac{1}{1+x^2} + \frac{1}{(1+x^2)^{3/2}} - \frac{1}{1+x^2}$$

$$RHS = \frac{1}{(1+x^2)^{3/2}} = \phi(x) = L.H.S$$

$\therefore$ $\phi(x)$ is a solution of (1). Proved.

(3) Show $\phi(x) = \frac{1}{\pi\sqrt{x}}$ is a solution of

$$\int_0^x \frac{\phi(\xi)}{\sqrt{x-\xi}}\,d\xi = 1$$

Solution. Given $\int_0^x \frac{\phi(\xi)}{\sqrt{x-\xi}}\,d\xi = 1$ —— (1)

To show $\phi(x) = \frac{1}{\pi\sqrt{x}}$ is a solution of (1)

$$LHS = \int_0^x \frac{1}{\sqrt{x-\xi}} \cdot \phi(\xi)\,d\xi$$

$$LHS = \int_0^x \frac{1}{\sqrt{x-\xi}} \cdot \frac{1}{\pi\sqrt{\xi}}\,d\xi$$

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$$LHS = \frac{1}{\pi}\int_0^x \frac{d\xi}{\sqrt{\xi x - \xi^2}}$$

Consider

$$-\xi^2 + \xi x$$
$$= -[\xi^2 - \xi x]$$
$$= -\left[\left(\xi - \frac{x}{2}\right)^2 - \frac{x^2}{4}\right]$$
$$= \frac{x^2}{4} - \left(\xi - \frac{x}{2}\right)^2$$

$$LHS = \frac{1}{\pi}\int_0^x \frac{d\xi}{\sqrt{\left(\frac{x}{2}\right)^2 - \left(\xi - \frac{x}{2}\right)^2}}$$

$$LHS = \frac{1}{\pi}\left[\sin^{-1}\left(\frac{\xi - \frac{x}{2}}{\frac{x}{2}}\right)\right]_0^x$$

$$\left[\because \int_0^x \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$LHS = \frac{1}{\pi}\left[\sin^{-1}\frac{x - \frac{x}{2}}{\frac{x}{2}} - \sin^{-1}\left(\frac{0 - \frac{x}{2}}{\frac{x}{2}}\right)\right]$$

$$LHS = \frac{1}{\pi}\left[\sin^{-1}(1) - \sin^{-1}(-1)\right]$$

$$LHS = \frac{1}{\pi}\left[\sin^{-1}(1) + \sin^{-1}(1)\right] = \frac{1}{\pi}\cdot 2\sin^{-1} 1$$

$$LHS = 2\cdot\frac{1}{\pi}\cdot\frac{\pi}{2} = 1 = RHS$$

$\therefore$ $\phi(x)$ is a solution of (1).

(4) Show $\phi(x) = (1-x)$ is a solution of

$$\int_0^x e^{x-\xi}\phi(\xi)\,d\xi = x$$

Solution. Given $\int_0^x e^{x-\xi}\phi(\xi)\,d\xi = x$ ——— (1)

To show $\phi(x) = (1-x)$ is a solution of (1).

$$LHS = \int_0^x e^{x-\xi}\phi(\xi)\,d\xi$$

$$LHS = e^x\int_0^x e^{-\xi}\cdot(1-\xi)\,d\xi$$

$$LHS = e^x\left[(1-\xi)\frac{e^{-\xi}}{-1}\Big|_0^x - \int_0^x (-1)\frac{e^{-\xi}}{-1}\,d\xi\right]$$

$$LHS = e^{x}\left[-(1-x)e^{-x} + (1-0)e^{-0} - \int_0^x e^{-\xi} d\xi\right]$$

$$LHS = e^{x}\left[-e^{-x} + xe^{-x} + 1 - \left(\frac{e^{-\xi}}{-1}\right)_0^x\right]$$

$$LHS = e^{x}\left[-e^{-x} + xe^{-x} + 1 + (e^{-x} - 1)\right]$$

$$LHS = e^{x}\left[-\cancel{e^{-x}} + xe^{-x} + 1 + \cancel{e^{-x}} - 1\right]$$

$$LHS = e^{x}\left[xe^{-x}\right] = x$$

$LHS = RHS$

which shows that $\phi(x)$ is a solution of (1)., Proved.

(5) Show that the function $\phi(x) = 1$ is a solution of

$$\phi(x) + \int_0^1 x(e^{x\xi} - 1)\phi(\xi)d\xi = e^{x} - x$$

Solution. Given $\phi(x) + \int_0^1 x(e^{x\xi} - 1)\phi(\xi)d\xi = (e^{x} - x)$ —— (1)

To show $\phi(x) = 1$ is a solution of (1)

$\therefore$ $LHS = \phi(x) + \int_0^1 x(e^{x\xi} - 1)\phi(\xi)d\xi$

$LHS = 1 + \int_0^1 x(e^{x\xi} - 1).1\, d\xi$ —— (2)

$LHS = 1 + \int_0^1 xe^{x\xi}d\xi - x\int_0^1 d\xi$ $\left[\because \phi(x) = 1 \;\&\; \phi(\xi) = 1\right]$

$LHS = 1 + \cancel{x}.\left[\frac{e^{x\xi}}{\cancel{x}}\right]_0^1 - x$ $\left[\int_0^1 d\xi = 1\right]$

$LHS = 1 + e^{x} - 1 - x$

$LHS = (e^{x} - x)$

$LHS = RHS$, which shows that $\phi(x) = 1$ is a solution of (1).

(6) Show that $\phi(x) = e^{x}\left[2x - \frac{2}{3}\right]$ is a solution of

$$\phi(x) + \lambda \int_0^1 e^{x-\xi}\phi(\xi)\,d\xi = 2xe^{x}, \ \lambda = 2.$$

Solution. Given

$$\phi(x) + 2\int_0^1 e^{x-\xi}\phi(\xi)\,d\xi = 2xe^{x} \quad \text{--- (1)}$$

$$[\because \lambda = 2]$$

To show $\phi(x) = e^{x}\left[2x - \frac{2}{3}\right]$ is a solution of (1).

$$LHS = e^{x}\left[2x - \tfrac{2}{3}\right] + 2\int_0^1 e^{x}\cdot e^{-\xi}\cdot e^{\xi}\left[2\xi - \tfrac{2}{3}\right]d\xi$$

$$LHS = e^{x}\left[2x - \tfrac{2}{3}\right] + 2e^{x}\int_0^1 \left(2\xi - \tfrac{2}{3}\right)d\xi$$

$$LHS = e^{x}\left[2x - \tfrac{2}{3}\right] + 2e^{x}\left[\left(\xi^{2} - \tfrac{2}{3}\xi\right)_0^1\right]$$

$$LHS = e^{x}\left[2x - \tfrac{2}{3}\right] + 2e^{x}\left[\left(1 - \tfrac{2}{3}\right) - (0-0)\right]$$

$$LHS = 2e^{x}\cdot x - \cancel{\tfrac{2}{3}e^{x}} + \cancel{\tfrac{2e^{x}}{3}} \qquad \left[\because \left(1 - \tfrac{2}{3}\right) = \tfrac{1}{3}\right]$$

$$LHS = 2xe^{x}$$

$LHS = RHS$, which shows that $\phi(x) = e^{x}\left(2x - \frac{2}{3}\right)$ is a solution of (1).

(7) Verify $\phi(x) = \frac{1}{2}$ is a solution of

$$\int_0^x \frac{\phi(\xi)}{\sqrt{x-\xi}}\,d\xi = \sqrt{x}$$

Solution. Given $\displaystyle\int_0^x \frac{\phi(\xi)}{\sqrt{x-\xi}}\,d\xi = \sqrt{x}$ --- (1)

To show $\phi(x) = \frac{1}{2}$ varifies (1).

$$LHS = \int_0^x \frac{\phi(\xi)}{\sqrt{x-\xi}}\,d\xi = \int_0^x \frac{\frac{1}{2}}{\sqrt{x-\xi}}\,d\xi \qquad x - \xi = t^{2}$$

$$LHS = \int \frac{1}{\cancel{2}}\cdot\frac{1}{\cancel{t}}\cdot -\cancel{2}\cancel{t}\,dt, \ t = \sqrt{x-\xi} \qquad -d\xi = 2t\,dt$$

$$LHS = -\int dt = -[t]_0^x = -\left[\sqrt{x-\xi}\right]_0^x = \sqrt{x}$$

$LHS = RHS$, which shows that $\phi(x) = \frac{1}{2}$ varifies (1).

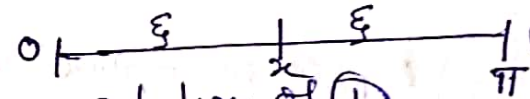
(8) Show that $\phi(x) = \cos 2x$ is a solution of the equation $\phi(x) = \cos x + 3\int_0^{\pi} K(x,\xi)\,\phi(\xi)\,d\xi$

where $K(x,\xi) = \begin{cases} -\sin x \cos\xi, & 0 \le x \le \xi \\ \cos x \sin\xi, & \xi \le x \le \pi \end{cases}$

Solution. Given

$$\phi(x) = \cos x + 3\int_0^{\pi} K(x,\xi)\,\phi(\xi)\,d\xi$$

where $K(x,\xi) = \begin{cases} -\sin x \cos\xi, & 0 \le x \le \xi \\ \cos x \sin\xi, & \xi \le x \le \pi \end{cases}$ (1)



To show $\phi(x) = \cos 2x$ is a solution of (1)

$$RHS = \cos x + 3\int_0^{\pi} K(x,\xi)\,\phi(\xi)\,d\xi$$

$$RHS = \cos x + 3\int_0^{x} K(x,\xi)\,\phi(\xi)\,d\xi + 3\int_x^{\pi} K(x,\xi)\,\phi(\xi)\,d\xi$$

$0 \le \xi \le x$ (Here) $\qquad x \le \xi \le \pi$ (Here)

$$RHS = \cos x + 3\int_0^{x} \cos x \sin\xi\,\phi(\xi)\,d\xi + 3\int_x^{\pi} -\sin x\cos\xi\,\phi(\xi)\,d\xi$$

$$RHS = \cos x + 3\int_0^{x} \cos x \sin\xi \cdot \cos 2\xi\,d\xi + 3\int_x^{\pi} -\sin x \cos\xi \cos 2\xi\,d\xi$$

$\left[\because \phi(x) = \cos 2x \Rightarrow \phi(\xi) = \cos 2\xi\right]$

$$RHS = \cos x + \frac{3\cos x}{2}\int_0^{x} 2\cos 2\xi \sin\xi\,d\xi + \frac{3 \cdot -\sin x}{2}\int_x^{\pi} 2\cos 2\xi \cos\xi\,d\xi$$

$$RHS = \cos x + \frac{3\cos x}{2}\int_0^{x} (\sin 3\xi - \sin\xi)\,d\xi + \frac{3 \cdot -\sin x}{2}\int_x^{\pi} (\cos 3\xi + \cos\xi)\,d\xi$$

$$RHS = \cos x + \frac{3\cos x}{2}\left[-\frac{\cos 3\xi}{3} + \cos\xi\right]_0^{x} + \frac{3 \cdot -\sin x}{2}\left[\frac{\sin 3\xi}{3} + \sin\xi\right]_x^{\pi}$$

$$RHS = \cos x + \frac{3}{2}\cos x\left[\left\{-\frac{\cos 3x}{3} + \cos x\right\} - \left\{-\frac{1}{3}+1\right\}\right]$$
$$+ \frac{3}{2}\sin x\left[\left(\frac{\sin 3\pi}{3} + \sin\pi\right) - \left(\frac{\sin 3x}{3} + \sin x\right)\right]$$

$$RHS = \cos x - \frac{1}{2}\cos x\cos 3x + \frac{3}{2}\cos^2 x + \left(\frac{3}{2}\cos x\right)\left(-\frac{2}{3}\right)$$
$$+\frac{3}{2}\sin x\left[0 - \frac{\sin 3x}{3} - \sin x\right] \qquad \left[\begin{array}{l}\sin 3\pi = 0\\ \sin\pi = 0\end{array}\right]$$

$$RHS = \cancel{\cos x} - \frac{1}{2}\cos x\cos 3x + \frac{3}{2}\cos^2 x - \cancel{\cos x} - \frac{1}{2}\sin x\sin 3x$$
$$-\frac{3}{2}\sin^2 x$$

$$RHS = -\frac{1}{2}\left[\cos x\cos 3x + \sin x\sin 3x\right] + \frac{3}{2}\left[\cos^2 x - \sin^2 x\right]$$

$$RHS = -\frac{1}{2}\cdot\cos 2x + \frac{3}{2}\cos 2x$$

$$RHS = \cos 2x$$

$RHS = \phi(x)$

$RHS = LHS$, which shows that $\phi(x)$ is a solution of (1)

(9) Show that $\phi(x) = \sin\left(\frac{\pi x}{2}\right)$ is a solution of

$$\phi(x) = \frac{1}{2}x + \frac{\pi^2}{4}\int_0^1 K(x,\xi)\,\phi(\xi)\,d\xi$$

, where $K(x,\xi) = \begin{cases}\frac{x}{2}(2-\xi), & 0 \le x \le \xi\\ \frac{\xi}{2}(2-x), & \xi \le x \le 1\end{cases}$

Solution. Given

$$\phi(x) = \frac{1}{2}x + \frac{\pi^2}{4}\int_0^1 K(x,\xi)\,\phi(\xi)\,d\xi$$

where $K(x,\xi) = \begin{cases}\frac{x}{2}(2-\xi), & 0 \le x \le \xi\\ \frac{\xi}{2}(2-x), & \xi \le x \le 1\end{cases}$ ——— (1)

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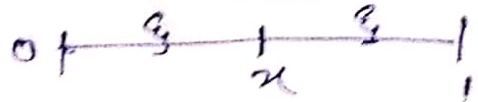
To show $\phi(x) = \sin\left(\frac{\pi x}{2}\right)$ is a solution of (1).



$$RHS = \frac{x}{2} + \frac{\pi^2}{4}\int_0^1 K(x,\xi)\phi(\xi)\,d\xi$$

$$RHS = \frac{x}{2} + \frac{\pi^2}{4}\int_0^x K(x,\xi)\phi(\xi)\,d\xi + \frac{\pi^2}{4}\int_x^1 K(x,\xi)\phi(\xi)\,d\xi$$

$$RHS = \frac{x}{2} + \frac{\pi^2}{4}\int_0^x \frac{\xi}{2}(2-x)\phi(\xi)\,d\xi + \frac{\pi^2}{4}\int_x^1 \frac{x}{2}(2-\xi)\phi(\xi)\,d\xi$$

$$RHS = \frac{x}{2} + \frac{\pi^2}{8}(2-x)\int_0^x \xi\,\phi(\xi)\,d\xi + \frac{\pi^2}{8}\cdot x\int_x^1 (2-\xi)\phi(\xi)\,d\xi$$

$$RHS = \frac{x}{2} + \frac{\pi^2}{8}(2-x)\int_0^x \xi\cdot\sin\left(\frac{\pi\xi}{2}\right)d\xi + \frac{\pi^2 x}{8}\int_x^1 (2-\xi)\sin\left(\frac{\pi\xi}{2}\right)d\xi$$

$$\left[\because \phi(x) = \sin\frac{\pi x}{2} \Rightarrow \phi(\xi) = \sin\frac{\pi\xi}{2}\right]$$

$$RHS = \frac{x}{2} + \frac{\pi^2}{8}(2-x)\left[(-\xi)\cdot\frac{2}{\pi}\cos\frac{\pi\xi}{2}\Big|_0^x - \int_0^x (1)(-)\cdot\frac{2}{\pi}\cos\frac{\pi\xi}{2}\,d\xi\right]$$

$$+ \frac{\pi^2}{8}x\left[(-)(2-\xi)\cdot\frac{2}{\pi}\cos\frac{\pi\xi}{2}\Big|_x^1 - \int_x^1 (-)(-)\frac{2}{\pi}\cos\frac{\pi\xi}{2}\,d\xi\right]$$

$$RHS = \frac{x}{2} + \frac{\pi^2}{8}(2-x)\left[(-x)\cdot\frac{2}{\pi}\cos\frac{\pi x}{2} + 0 + \frac{2}{\pi}\int_0^x \cos\frac{\pi\xi}{2}\,d\xi\right]$$

$$+ \frac{\pi^2}{8}x\left[(-)(1)\cdot\frac{2}{\pi}\cos\frac{\pi}{2} + (2-x)\cdot\frac{2}{\pi}\cos\frac{\pi x}{2} - \frac{2}{\pi}\int_x^1 \cos\frac{\pi\xi}{2}\,d\xi\right]$$

$$RHS = \frac{x}{2} - \cancel{\frac{\pi}{4}(2x-x^2)\cos\frac{\pi x}{2}} + \frac{\pi}{4}(2-x)\cdot\frac{2}{\pi}\left[\sin\frac{\pi\xi}{2}\right]_0^x$$

$$+ \cancel{\frac{\pi^2}{8}(2x-x^2)\cdot\frac{2}{\pi}\cos\frac{\pi x}{2}} - \frac{\pi}{4}x\left[\frac{2}{\pi}\sin\frac{\pi\xi}{2}\right]_x^1$$

$$RHS = \frac{x}{2} + \frac{2-x}{2}\left[-\sin\frac{\pi x}{2} - 0\right]$$
$$- \frac{\pi x}{4}\cdot\frac{2}{\pi}\left[-\sin\frac{\pi}{2} - \sin\frac{\pi x}{2}\right]$$
$$= \frac{x}{2} + \sin\frac{\pi x}{2} - \cancel{\frac{x}{2}\sin\frac{\pi x}{2}} - \frac{x}{2} + \cancel{\frac{x}{2}\sin\frac{\pi x}{2}}$$

$$= \sin\frac{\pi x}{2}$$
$$= LHS$$

RHS $= \phi(x)$, which shows that $\phi(x)$ is a solution of (1).

(10). Verify that $\phi(x) = \dfrac{x}{(1+x^2)^{5/2}}$ is the solution of
$$\phi(x) = \frac{3x+2x^3}{3(1+x^2)^2} - \int_0^x \frac{3x+2x^3-\xi}{(1+x^2)^2}\phi(\xi)\,d\xi$$

Solution. Given
$$\phi(x) = \frac{3x+2x^3}{3(1+x^2)^2} - \int_0^x \frac{3x+2x^3-\xi}{(1+x^2)^2}\cdot\phi(\xi)\,d\xi \quad \text{---} (1)$$

To verify $\phi(x) = \dfrac{x}{(1+x^2)^{5/2}}$ is a solution of (1)
$$RHS \text{ of } (1) = \frac{3x+2x^3}{3(1+x^2)^2} - \int_0^x \frac{3x+2x^3-\xi}{(1+x^2)^2}\phi(\xi)\,d\xi$$
$$RHS = \frac{3x+2x^3}{3(1+x^2)^2} - \frac{1}{(1+x^2)^2}\int_0^x (3x+2x^3-\xi)\cdot\frac{\xi}{(1+\xi^2)^{5/2}}\,d\xi$$

$$\left[\because \phi(x) = \frac{x}{(1+x^2)^{5/2}} \Rightarrow \phi(\xi) = \frac{\xi}{(1+\xi^2)^{5/2}}\right]$$

$$RHS = \frac{3x+2x^3}{3(1+x^2)^2} - \frac{1.(3x+2x^3)}{(1+x^2)^2}\int_0^x \frac{\xi}{(1+\xi^2)^{5/2}}\,d\xi$$

$1+\xi^2 = t^2$

$\xi\, d\xi = t\, dt$

$$+ \frac{1}{(1+x^2)^2}\int_0^x \frac{\xi^2}{(1+\xi^2)^{5/2}}\,d\xi$$

$$RHS = \frac{3x+2x^3}{3(1+x^2)^2} - \frac{3x+2x^3}{(1+x^2)^2}\int \frac{t\,dt}{t^5} \quad (t=\sqrt{1+\xi^2})$$

$$+ \frac{1}{(1+x^2)^2}\int_0^x \frac{\xi^2}{(1+\xi^2)^{5/2}}\,d\xi$$

$$RHS = \frac{3x+2x^3}{3(1+x^2)^2} - \frac{3x+2x^3}{(1+x^2)^2}\left(-\frac{1}{3}\right)\left[\frac{1}{t^3}\right]_0^x, \quad t=\sqrt{1+\xi^2}$$

$$+ \frac{1}{(1+x^2)^2}\int_0^x \frac{\xi^2}{(1+\xi^2)^{5/2}}\,d\xi$$

$$= \frac{3x+2x^3}{3(1+x^2)^2} + \frac{1}{3}\cdot\frac{3x+2x^3}{(1+x^2)^2}\left[\frac{1}{(1+\xi^2)^{3/2}}\right]_0^x$$

$$+ \frac{1}{(1+x^2)^2}\int_0^x \frac{\xi^2}{(1+\xi^2)^{5/2}}\,d\xi$$

$$RHS = \frac{3x+2x^3}{3(1+x^2)^2} + \frac{1}{3}\frac{3x+2x^3}{(1+x^2)^2}\left[\frac{1}{(1+x^2)^{3/2}} - 1\right] + I_1 \quad \text{--- (2)}$$

where
$$I_1 = \frac{1}{(1+x^2)^2}\int_0^x \frac{\xi^2}{(1+\xi^2)^{5/2}}\,d\xi$$

put $\xi = \tan\theta$

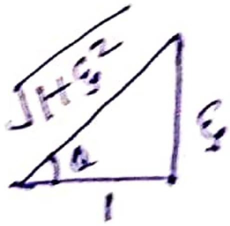
$d\xi = \sec^2\theta\, d\theta$

$$I_1 = \frac{1}{(1+x^2)^2}\int \frac{\tan^2\theta}{(1+\tan^2\theta)^{5/2}}\cdot\sec^2\theta\, d\theta$$

$$I_1 = \frac{1}{(1+x^2)^2}\int \frac{\tan^2\theta}{\sec^5\theta}\cdot \sec^2\theta\, d\theta, \quad \text{where } \tan\theta = \xi$$

$$I_1 = \frac{1}{(1+x^2)^2}\int \frac{\sin^2\theta}{\cos^2\theta \cdot \sec^2\theta \cdot \sec\theta}\, d\theta$$

$$I_1 = \frac{1}{(1+x^2)^2}\int_{\xi=0}^{\xi=x} \sin^2\theta \cdot \cos\theta\, d\theta$$



$$I_1 = \frac{1}{(1+x^2)^2}\cdot \frac{1}{3}(\sin\theta)^3$$

$$\sin\theta = \frac{\xi}{\sqrt{1+\xi^2}}$$

$$I_1 = \frac{1}{3}\cdot\frac{1}{(1+x^2)^2}\left[\frac{\xi^3}{(1+\xi^2)^{3/2}}\right]_{\xi=0}^{\xi=x}$$

$$I_1 = \frac{1}{3(1+x^2)}\left[\frac{x^3}{(1+x^2)^{3/2}} - 0\right] \quad \text{—— (3)}$$

Putting the value of I_1 into (2), we have

$$RHS = \frac{3x+2x^3}{3(1+x^2)^2} + \frac{1}{3}\,\frac{3x+2x^3}{(1+x^2)^2(1+x^2)^{3/2}} - \frac{1}{3}\cdot\frac{3x+2x^3}{(1+x^2)^2}$$

$$+ \frac{x^3}{3(1+x^2)^2(1+x^2)^{3/2}}$$

$$RHS = \frac{1}{3(1+x^2)^2}\left[\cancel{(3x+2x^3)} + \frac{3x+2x^3}{(1+x^2)^{3/2}} - \cancel{(3x+2x^3)} + \frac{x^3}{(1+x^2)^{3/2}}\right]$$

$$RHS = \frac{1}{3(1+x^2)^2}\left[\frac{3x+3x^3}{(1+x^2)^{3/2}}\right] = \frac{3x\cancel{(1+x^2)}}{3\cancel{(1+x^2)}(1+x^2)(1+x^2)^{3/2}}$$

$$RHS = \frac{x}{(1+x^2)^{5/2}} = f(x) = LHS,$$ which shows that $f(x)$ is a solution of (1).

Problems for Practice

(1) Show $\phi(x) = \left(x - \frac{1}{6}x^3\right)$ is a solution of

$$\phi(x) = x - \int_0^x \sinh(x-\xi)\,\phi(\xi)\,d\xi \quad \text{--- (1)}$$

Hint:

$$RHS = x - \int_0^x \left(\sinh x \cosh\xi - \cosh x \sinh\xi\right)\left(\xi - \frac{\xi^3}{6}\right)d\xi \quad \text{--- (2)}$$

$$RHS = x - \sinh x \cdot I_1 + \cosh x \cdot I_2$$

where $I_1 = \int_0^x \left(\xi - \frac{\xi^3}{6}\right)\cosh\xi\,d\xi$, $I_2 = \int_0^x \left(\xi - \frac{\xi^3}{6}\right)\sinh\xi\,d\xi$

$$I_1 = \left(\xi - \frac{\xi^3}{6}\right)\cdot\sinh\xi\Big|_0^x - \int_0^x \left(1 - \frac{\xi^2}{2}\right)\sinh\xi\,d\xi$$

$$= \left(x - \frac{x^3}{6}\right)\sinh x - 0 - \left[\left(1 - \frac{\xi^2}{2}\right)\cdot\cosh\xi\Big|_0^x - \int_0^x (0 - \xi)\cosh\xi\,d\xi\right]$$

$$= \left(x - \frac{x^3}{6}\right)\sinh x - \left(1 - \frac{x^2}{2}\right)\cosh x + 1 - \int_0^x \xi\cosh\xi\,d\xi$$

$$= \left(x - \frac{x^3}{6}\right)\sinh x - \cosh x + \frac{x^2}{2}\cosh x + 1 - \left[\xi\cdot\sinh\xi\Big|_0^x - \int_0^x \sinh\xi\,d\xi\right]$$

$$= x\sinh x - \frac{x^3}{6}\sinh x - \cosh x + \frac{x^2}{2}\cosh x + 1 - x\sinh x + 0 + \left[\cosh\xi\right]_0^x$$

$$= \underline{x\sinh x} - \frac{x^3}{6}\sinh x - \underline{\cosh x} + \frac{x^2}{2}\cosh x + 1 - \underline{x\sinh x} + \underline{\cosh x} - 1$$

$$I_1 = \left(-\frac{x^3}{6}\sinh x + \frac{x^2}{2}\cosh x\right)$$

$$I_2 = \int_0^x \left(\xi - \frac{\xi^3}{6}\right)\sinh\xi\,d\xi$$

$$I_2 = \left(\xi - \frac{\xi^3}{6}\right)\cosh\xi\Big|_0^x - \int_0^x \left(1 - \frac{\xi^2}{2}\right)\cosh\xi\,d\xi$$

$$I_2 = \left(x - \frac{x^3}{6}\right)\cosh x - 0 - \left[\left(1 - \frac{\xi^2}{2}\right)\sinh\xi\Big|_0^x - \int_0^x (0 - \xi)\sinh\xi\,d\xi\right]$$

$$I_2 = \left(x-\frac{x^3}{6}\right)\cosh x - \left(1-\frac{x^2}{2}\right)\sinh x + 0 - \int_0^x \xi \sinh\xi \, d\xi$$

$$= \left(x-\frac{x^3}{6}\right)\cosh x - \left(1-\frac{x^2}{2}\right)\sinh x - \left[\xi.\cosh\xi\Big|_0^x - \int_0^x \cosh\xi \, d\xi\right]$$

$$= \left(x-\frac{x^3}{6}\right)\cosh x - \left(1-\frac{x^2}{2}\right)\sinh x - x\cosh x + 0 + \int_0^x \cosh\xi \, d\xi$$

$$= \cancel{x\cosh x} - \frac{x^3}{6}\cosh x - \sinh x + \frac{x^2}{2}\sinh x - \cancel{x\cosh x}$$
$$+ \left[\sinh\xi\right]_0^x$$

$$= -\frac{x^3}{6}\cosh x - \cancel{\sinh x} + \frac{x^2}{2}\sinh x + \cancel{\sinh x} - 0$$

$$I_2 = -\frac{x^3}{6}\cosh x + \frac{x^2}{2}\sinh x$$

Putting the values of I_1 and I_2 into (2), we have

$$RHS = x - \sinh x . I_1 + \cosh x . I_2$$

$$= x - \sinh x\left[-\frac{x^3}{6}\sinh x + \frac{x^2}{2}\cosh x\right]$$
$$+ \cosh x\left[-\frac{x^3}{6}\cosh x + \frac{x^2}{2}\sinh x\right]$$

$$= x + \frac{x^3}{6}\sinh^2 x - \cancel{\frac{x^2}{2}\sinh x\cosh x} - \frac{x^3}{6}\cosh^2 x + \cancel{\frac{x^2}{2}\sinh x\cosh x}$$

$$= x - \frac{x^3}{6}\left[\cosh^2 x - \sinh^2 x\right]$$

$$= \left(x - \frac{x^3}{6}\right) \qquad \left[\because \cosh^2 x - \sinh^2 x = 1\right]$$

$$= \phi(x)$$
$$= LHS$$

which shows that $\phi(x)$ is a solution of (1).

(2) $\phi(x) = 3$ is a solution of $x^3 = \int_0^x (x-\xi)^2 \phi(\xi) d\xi$

(3) $\phi(x) = \cos x$ is a solution of

$$\phi(x) = \sin x + \int_0^{\pi} (x^2+\xi)\cos\xi \, \phi(\xi) d\xi.$$