SUBJECT: MATHEMATICS

PAPER: ADVANCED MATHEMATICAL METHODS
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Symmetric Kernel. A kernel $K(x,\xi)$ is called symmetric (or Hermitian) kernel if

$$K(x,\xi) = K(\xi, x)$$

Note: $\overline{K(\xi,x)} = K(x,\xi)$ if $K(\xi,x)$ is complex

Theorem. 1. All iterated kernels of a symmetric kernel are also symmetric.

Proof. The iterated kernels are defined as

$$K_1(x,\xi) = K(x,\xi)$$
$$K_2(x,\xi) = \int_a^b K_1(x,z)\, K_1(z,\xi)\, dz$$
$$K_3(x,\xi) = \int_a^b K_1(x,z)\, K_2(z,\xi)\, dz$$
$$K_4(x,\xi) = \int_a^b K_1(x,z)\, K_3(z,\xi)\, dz$$

— — — — etc.

We know that

$$K_n(\xi,x) = \int_a^b\int_a^b \cdots \int_a^b K(\xi,z_1)\, K(z_1,z_2)\, K(z_2,z_3) \cdots K(z_{n-1},x) \times dz_1\, dz_2 \cdots dz_{n-1}$$

$$= \int_a^b\int_a^b \cdots \int_a^b K(z_1,\xi)\, K(z_2,z_1)\, K(z_3,z_2) \cdots K(x,z_{n-1}) \times dz_1\, dz_2 \cdots dz_{n-1}$$

$$= \int_a^b\int_a^b \cdots \int_a^b K(x,z_{n-1})\, K(z_{n-1},z_{n-2}) \cdots K(z_2,z_1)\, K(z_1,\xi) \times dz_1\, dz_2 \cdots dz_{n-1}$$

substituting $z_{n-1} = z_1,\ z_{n-2} = z_2,\ z_{n-3} = z_3 \ldots$

$$= \int_a^b\int_a^b \cdots \int_a^b K(x,z_1)\, K(z_1,z_2) \cdots K(z_{n-1},\xi)\, dz_1\, dz_2 \cdots dz_{n-1}$$

$K_n(\xi,x) = K_n(x,\xi)$

Thus The kernel K_n is also symmetric. Hence by

Induction we conclude that iterated kernel of a symmetric kernel is symmetric.

Theorem 02. Let $K(x,\xi)$ is symmetric kernel. Let λ_m and $\lambda_n\,(\because \lambda_m \neq \lambda_n)$ be any two different eigen values having their corresponding eigen functions $\phi_m(x)$ and $\phi_n(x)$ respectively then

$$\int_a^b \phi_m(x)\,\phi_n(x)\,dx = 0 \quad \text{for } m \neq n.$$

Proof. Let the homogeneous Fredholm integral equation is given by

$$\phi(x) = \lambda \int_a^b K(x,\xi)\,\phi(\xi)\,d\xi \qquad \text{--- (1)}$$

where $K(x,\xi) = K(\xi,x)$ --- (2)

Let $\phi_m(x), \phi_n(x)$ be the eigen functions corresponding to their eigen values λ_m and λ_n respectively, then we have

$$\phi_m(x) = \lambda_m \int_a^b K(x,\xi)\,\phi_m(\xi)\,d\xi \qquad \text{--- (3)}$$

$$\phi_n(x) = \lambda_n \int_a^b K(x,\xi)\,\phi_n(\xi)\,d\xi \qquad \text{--- (4)}$$

multiplying (3) by $\phi_n(x)$ to both sides and integrating w.r.to x from $x=a$ to $x=b$, we have

$$\int_a^b \phi_m(x)\,\phi_n(x)\,dx = \int_a^b \lambda_m\,\phi_n(x)\left\{\int_a^b K(x,\xi)\,\phi_m(\xi)\,d\xi\right\}dx$$

Changing the order of integration, we have

$$\int_a^b \phi_m(x)\,\phi_n(x)\,dx = \int_a^b \lambda_m\,\phi_m(\xi)\left\{\int_a^b \phi_n(x)\,K(x,\xi)\,dx\right\}d\xi$$

$$\int_a^b \phi_m(x)\phi_n(x)\,dx = \int_a^b \lambda_m\,\phi_m(\xi)\left\{\int_a^b K(\xi,x)\,\phi_n(x)\,dx\right\}d\xi \qquad \text{--- (5)}$$

$(\because K(x,\xi) = K(\xi, \ldots$

Now, from (4) $x \rightleftarrows \xi$ (i.e. interchanging x & ξ in (4))

$$\phi_n(\xi) = \lambda_n \int_a^b K(\xi, x)\, \phi_n(x)\,dx$$

$$\Rightarrow \int_a^b K(\xi, x)\, \phi_n(x)\,dx = \frac{1}{\lambda_n}\phi_n(\xi) \quad \text{——— (6)}$$

Equ's (5) & (6) $\Rightarrow$

$$\int_a^b \phi_m(x)\,\phi_n(x)\,dx = \int_a^b \lambda_m \phi_m(\xi)\cdot \frac{1}{\lambda_n}\phi_n(\xi)\,d\xi$$

$$\Rightarrow \int_a^b \phi_m(x)\,\phi_n(x)\,dx = \frac{\lambda_m}{\lambda_n}\int_a^b \phi_m(x)\,\phi_n(x)\,dx$$

put x for ξ

$$\Rightarrow \left(1 - \frac{\lambda_m}{\lambda_n}\right)\int_a^b \phi_m(x)\,\phi_n(x)\,dx = 0$$

$$\Rightarrow \left(\frac{\lambda_n - \lambda_m}{\lambda_n}\right)\int_a^b \phi_m(x)\,\phi_n(x)\,dx = 0 \quad \text{——— (7)}$$

$$\Rightarrow \int_a^b \phi_m(x)\,\phi_n(x)\,dx = 0 \quad \text{as } \lambda_m \neq \lambda_n.$$

$\Rightarrow$ $\phi_m(x)$ and $\phi_n(x)$ are orthogonal.

Theorem 03. Prove that eigen values of a symmetric kernel are real.

Proof. Suppose

$$\phi(x) = \lambda \int_a^b K(x, \xi)\,\phi(\xi)\,d\xi \quad \text{be a homogeneous} \quad \text{——— (1)}$$

Fredholm integral equation with symmetric kernel.

Let $\lambda_m = \alpha + i\beta$ and $\bar{\lambda}_m = \alpha - i\beta$ be the eigen values corresponding to their eigen functions $\phi_m(x) = \psi_1 + i\psi_2$ and $\bar{\phi}_m(x) = \bar{\psi}_1 - i\bar{\psi}_2$ respectively, then we have

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$$\phi_m(x) = \lambda_m \int_a^b K(x,\xi)\,\phi_m(\xi)\,d\xi \quad \text{--- (2)}$$

$$\bar{\phi}_m(x) = \bar{\lambda}_m \int_a^b K(x,\xi)\,\bar{\phi}_m(\xi)\,d\xi \quad \text{--- (3)}$$

multiplying equation (2) by $\bar{\phi}_m(x)$ to both sides of (2) and integrating with respect to x from $x=a$ to $x=b$.

$$\int_a^b \phi_m(x)\bar{\phi}_m(x)\,dx = \int_a^b \lambda_m \bar{\phi}_m(x)\left\{\int_a^b K(x,\xi)\,\phi_m(\xi)\,d\xi\right\}dx$$

Interchanging the order of integration we have

$$\int_a^b \phi_m(x)\bar{\phi}_m(x)\,dx = \int_a^b \lambda_m \phi_m(\xi)\left[\int_a^b \bar{\phi}_m(x)\,K(x,\xi)\,dx\right]d\xi$$

$$\int_a^b \phi_m(x)\bar{\phi}_m(x)\,dx = \int_a^b \lambda_m \phi_m(\xi)\left[\int_a^b K(\xi,x)\,\bar{\phi}_m(x)\,dx\right]d\xi \quad \text{--- (4)}$$

Now interchanging x & ξ in (3) we have

$$\bar{\phi}_m(\xi) = \bar{\lambda}_m \int_a^b K(\xi,x)\,\bar{\phi}_m(x)\,dx$$

$$\Rightarrow \int_a^b K(\xi,x)\,\bar{\phi}_m(x)\,dx = \frac{1}{\bar{\lambda}_m}\bar{\phi}_m(\xi) \quad \text{--- (5)}$$

Eq^{u}s (4) and (5) $\Rightarrow$

$$\int_a^b \phi_m(x)\bar{\phi}_m(x)\,dx = \int_a^b \lambda_m \phi_m(\xi)\cdot\frac{1}{\bar{\lambda}_m}\bar{\phi}_m(\xi)\,d\xi$$

$$= \frac{\lambda_m}{\bar{\lambda}_m}\int_a^b \phi_m(x)\bar{\phi}_m(x)\,dx$$

(put x for ξ)

$$\Rightarrow \left(1 - \frac{\lambda_m}{\bar{\lambda}_m}\right)\int_a^b \phi_m(x)\bar{\phi}_m(x)\,dx = 0$$

$$\Rightarrow (\bar{\lambda}_m - \lambda_m)\int_a^b \phi_m(x)\bar{\phi}_m(x)\,dx = 0$$

$$\Rightarrow (\alpha + i\beta - \alpha + i\beta)\int_a^b (\psi_1\bar{\psi}_1 + \psi_2\bar{\psi}_2)\,dx = 0$$

$$\Rightarrow 2i\beta = 0 \text{ as } \int_a^b (\psi_1\bar{\psi}_1 + \psi_2\bar{\psi}_2)\,dx \neq 0.$$

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$\Rightarrow \beta = 0$

i) $\lambda_m = \alpha + i\beta$

$\lambda_m = \alpha$

$\Rightarrow \lambda_m$ is real

$\Rightarrow \bar{\lambda}_m$ is real

$\Rightarrow$ Eigen values of a symmetric kernel are real.

Theorem 04. If $K(x, \xi)$ is real and symmetric and identically not equal to zero then all of the characteristic constants are real.

Proof. Let $\phi(x) = \lambda_0 \int_a^b K(x, \xi)\, \phi(\xi)\, d\xi$ ——— (1)

be homogeneous equation with characteristic constant λ_0 having characteristic function $\phi(x)$. where $\phi(x)$ is real

Suppose if possible λ_0 is not real

Let $\lambda_0 = x + i.y,\ y \neq 0$ then (1) $\Rightarrow$

$\phi(x) = (x + iy) \int_a^b K(x, \xi)\, \phi(\xi)\, d\xi$

$\phi(x) = x \int_a^b K(x, \xi)\, \phi(\xi)\, d\xi + iy \int_a^b K(x, \xi)\, \phi(\xi)\, d\xi$ ——(2)

comparing real and imaginary parts, we have

——— (3)

$\phi(x) = x \int_a^b K(x, \xi)\, \phi(\xi)\, d\xi$

——(4)

$0 = y \int_a^b K(x, \xi)\, \phi(\xi)\, d\xi$

from (4) $\int_a^b K(x, \xi)\, \phi(\xi)\, d\xi = 0$

$\Rightarrow \phi(x) = 0$ which is a contradiction as

$\phi(x)$ is real $\Rightarrow \phi(x)$ can not be real

Suppose $\phi(x) = p(x) + i\, q(x)$ ——— (5)

So eqn's (1) & (5) $\Rightarrow$

$$p(x)+i\,q(x)=(x+iy)\int_a^b K(x,\xi)\{p(\xi)+i\,q(\xi)\}\,d\xi$$

Comparing real and imaginary parts, we have

$$p(x)=x\int_a^b K(x,\xi)p(\xi)\,d\xi-y\int_a^b K(x,\xi)q(\xi)\,d\xi \quad (6)$$

$$q(x)=x\int_a^b K(x,\xi)q(\xi)\,d\xi+y\int_a^b K(x,\xi)p(\xi)\,d\xi \quad (7)$$

multiplying (7) by $(-i)$, we have

$$-i\,q(x)=-xi\int_a^b K(x,\xi)q(\xi)\,d\xi-iy\int_a^b K(x,\xi)p(\xi)\,d\xi \quad (8)$$

Adding (6) & (8), we have

$$p(x)-i\,q(x)=(x-iy)\int_a^b K(x,\xi)p(\xi)\,d\xi-ix\int_a^b K(x,\xi)q(\xi)\,d\xi-y\int_a^b K(x,\xi)q(\xi)\,d\xi$$

$$=(x-iy)\int_a^b K(x,\xi)p(\xi)\,d\xi-i\left[x\int_a^b K(x,\xi)q(\xi)\,d\xi-iy\int_a^b K(x,\xi)q(\xi)\,d\xi\right]$$

$$=(x-iy)\int_a^b K(x,\xi)p(\xi)\,d\xi+x\int_a^b K(x,\xi)[(-i)q(\xi)]\,d\xi-iy\int_a^b K(x,\xi)[-i]q(\xi)\,d\xi \quad [\because i^2=-1]$$

$$=(x-iy)\int_a^b K(x,\xi)p(\xi)\,d\xi+(x-iy)\int_a^b K(x,\xi)[-iq(\xi)]\,d\xi$$

$$p(x)-iq(x)=(x-iy)\int_a^b K(x,\xi)[p(\xi)-iq(\xi)]\,d\xi$$

This shows $p(x)-iq(x)$ is also a characteristic function $[\text{where } \overline{\phi(x)}=p(x)-iq(x)]$ corresponding to $\overline{\lambda_0}=(x-iy)$

Since $\lambda_0\neq\overline{\lambda_0}\Rightarrow\int_a^b \phi(x)\overline{\phi(x)}\,dx=0$

$$\Rightarrow \int_a^b (p+iq)(p-iq)\,dx = 0$$

$$\Rightarrow \int_a^b [p^2(x) + q^2(x)]\,dx = 0$$

which is again a contradiction

$\therefore$ λ_0 can not be of the form $\lambda_0 = x+iy$

$\Rightarrow$ λ_0 must be real.

Hilbert Schmidt theorem

suppose $\phi(x) = f(x) + \lambda \int_a^b K(x,\xi)\phi(\xi)\,d\xi$ — (1)

be any fredholm integral equation.

let homogeneous equation of (1) is

$$\phi(x) = \lambda \int_a^b K(x,\xi)\,\phi(\xi)\,d\xi \qquad \text{— (2)}$$

let λ_n be the eigen values of (2) having their corresponding eigen values $\phi_n(x)$ then the solution of (1) is given by

$$\phi(x) = f(x) + \lambda \sum_{n=1}^{\infty} \frac{1}{\lambda_n - \lambda} F_n \Phi_n(x) \qquad \text{— (3)}$$

where $F_n = \int_a^b f(x)\Phi_n(x)\,dx$

$$\Phi_n(x) = \frac{\phi_n(x)}{\|\phi_n(x)\|}, \quad \|\phi_n(x)\| = \left[\int_a^b \phi_n^2(x)\,dx\right]^{\frac{1}{2}}$$

Equ (3) has three cases

Case I. if $\lambda \neq \lambda_n$, $F_n \neq 0$ then equation (3) has unique solution.

Case II. If $\lambda = \lambda_n$ and $F_n = 0$

then equation (1) has infinitely many solutions

as $\phi(x) = f(x) + c \cdot \Phi_n(x)$, c is arbitrary.

Case III if $\lambda = \lambda_n$ and $F_n \neq 0$

Then equation (1) has no solution.

Example.

① Using Hilbert schmidt theorem, solve

$$\phi(x) = f(x) + \lambda \int_a^b K(x,\xi)\phi(\xi)\,d\xi$$

where $K(x,\xi) = K(x) \cdot K(\xi)$

Solution. Given $\phi(x) = f(x) + \lambda \int_a^b K(x,\xi)\,\phi(\xi)\,d\xi$ —— ①

and $K(x,\xi) = K(x)\,K(\xi)$ —— ②

The Homogeneous equation of ① is

$$\phi(x) = \lambda \int_a^b K(x,\xi)\phi(\xi)\,d\xi \quad \text{—— ③}$$

$$\Rightarrow \phi(x) = \lambda \int_a^b K(x)\,K(\xi)\,\phi(\xi)\,d\xi \quad \text{; from (2)}$$

$$\Rightarrow \phi(x) = \lambda K(x) \cdot c_1 \quad \text{—— (4)}$$

$$c_1 = \int_a^b K(\xi)\,\phi(\xi)\,d\xi \quad \text{—— (5)}$$

from (4) $\phi(\xi) = \lambda c_1 K(\xi)$ —— (6)

Equ's (5) & ⑥ $\Rightarrow c_1 = \int_a^b K(\xi)\,\lambda c_1\,K(\xi)\,d\xi$

$$\Rightarrow c_1 = \int_a^b \lambda c_1 K^2(\xi)\,d\xi$$

$$\Rightarrow c_1\left[1-\lambda\int_a^b K^2(\xi)\,d\xi\right]=0$$

$$\Rightarrow 1-\lambda\int_a^b K^2(\xi)\,d\xi=0 \quad \text{if } c_1\neq 0.$$

$$\Rightarrow \lambda=\frac{1}{\int_a^b K^2(\xi)\,d\xi}=\lambda_1, \text{ say}$$

$\therefore$ Eqn (4) $\Rightarrow$

$$\phi(x)=\lambda K(x)\,c_1$$

For eigen value $\lambda=\lambda_1$, corresponding eigen function is given by

$$\phi_1(x)=\lambda_1 K(x)\cdot c_1$$

$$\Rightarrow \phi_1(x)=\frac{K(x)}{\int_a^b K^2(\xi)\,d\xi}\cdot c_1=\left[\frac{c_1}{\int_a^b K^2(\xi)\,d\xi}\right]\cdot K(x)$$

Take $\frac{c_1}{\int_a^b K^2(\xi)\,d\xi}=1$

$$\phi_1(x)=K(x)$$

$$\therefore \bar{\Phi}_1(x)=\frac{\phi_1(x)}{\|\phi_1(x)\|}=\frac{K(x)}{\left[\int_a^b K^2(\xi)\,d\xi\right]^{\frac{1}{2}}}$$

Also $F_1=\int_a^b f(x)\,\bar{\Phi}_1(x)\,dx$

$$F_1=\int_a^b f(x)\cdot\frac{K(x)}{\left[\int_a^b K^2(\xi)\,d\xi\right]^{\frac{1}{2}}}\cdot dx$$

The integral equation (1) will possess unique solution (Let $\lambda\neq\lambda_1$)

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$$\phi(x) = f(x) + \frac{\lambda}{\lambda_1 - \lambda} F_1 \cdot \Phi_1(x)$$

$$\Rightarrow \phi(x) = f(x) + \frac{\lambda}{\left[\frac{1}{\int_a^b K^2(\xi)d\xi} - \lambda\right]} \cdot \frac{\int_a^b f(x) K(x)\, dx}{\left[\int_a^b K^2(\xi)d\xi\right]^{\frac{1}{2}}} \times \frac{K(x)}{\left[\int_a^b K^2(\xi)d\xi\right]^{\frac{1}{2}}}$$

$$\phi(x) = f(x) + \frac{\lambda \int_a^b K^2(\xi)d\xi}{1 - \lambda\int_a^b K^2(\xi)d\xi} \times \frac{K(x) \cdot \int_a^b f(x) K(x)\, dx}{\int_a^b K^2(\xi)d\xi}$$

$$\phi(x) = f(x) + \frac{\lambda K(x) \int_a^b f(x) K(x)\, dx}{1 - \lambda \int_a^b K^2(\xi)d\xi}$$

<u>case II</u> let $\lambda = \lambda_1$ then

$$F_1 = \int_a^b f(x) \Phi_1(x)\, dx$$

$$F_1 = \int_a^b f(x) \cdot \frac{K(x)}{\left[\int_a^b K^2(\xi)d\xi\right]^{\frac{1}{2}}} dx \neq 0$$

(clearly) as $f(x) \neq 0$, $K(x) \neq 0$.

$\therefore$ the given integral equation has no solution.

<u>Case III</u> Suppose $f(x) \perp \Phi_1(x)$ i.e $\int_a^b f(x)\Phi_1(x)\,dx = 0$

i.e. $F_1 = 0$ then the integral eqn (i) possesses infinitely many solutions given by

$$\phi(x) = f(x) + c\,\Phi_1(x)$$

where c is arbitrary & $\Phi_1(x) = \frac{K(x)}{\left[\int_a^b K^2(\xi)d\xi\right]^{\frac{1}{2}}}$

(2) Using Hilbert-Schmidt theorem, solve ——— (1)

$$\phi(x) = (x+1)^2 + \int_{-1}^{1} (x\xi + x^2\xi^2)\,\phi(\xi)\,d\xi$$ ——— (2)

Solⁿ. Given $\phi(x) = (x+1)^2 + \int_{-1}^{1} (x\xi + x^2\xi^2)\,\phi(\xi)\,d\xi$

Here $\lambda = 1$

The homogeneous equation of (2) is given by ——— (3)

$$\phi(x) = \lambda \int_{-1}^{1} (x\xi + x^2\xi^2)\,\phi(\xi)\,d\xi$$ ——— (4)

$$\Rightarrow \quad \phi(x) = \lambda x c_1 + \lambda x^2 c_2$$ ——— (5)

where

$$c_1 = \int_{-1}^{1} \xi\,\phi(\xi)\,d\xi$$ ——— (6)

$$c_2 = \int_{-1}^{1} \xi^2\,\phi(\xi)\,d\xi$$ ——— (7)

From (4),

$$\phi(\xi) = (\lambda c_1 \xi + \lambda c_2 \xi^2)$$

Eqⁿ's (5) & (7) $\Rightarrow$

$$c_1 = \int_{-1}^{1} \xi[\lambda c_1 \xi + \lambda c_2 \xi^2]\,d\xi$$

$$= \lambda c_1 \int_{-1}^{1} \xi^2\,d\xi + \lambda c_2 \int_{-1}^{1} \xi^3\,d\xi$$

$$c_1 = \frac{2\lambda c_1}{3} + 0.c_2$$ ——— (8)

$$\Rightarrow \left(1 - \frac{2\lambda}{3}\right)c_1 + 0c_2 = 0$$

Equations (6) and (7) $\Rightarrow$

$$c_2 = \int_{-1}^{1} \xi^2[\lambda c_1 \xi + \lambda c_2 \xi^2)\,d\xi$$

$$= \lambda c_1 \int_{-1}^{1} \xi^3\,d\xi + \lambda c_2 \int_{-1}^{1} \xi^4\,d\xi$$

$$c_2 = 0.c_1 + \frac{2\lambda}{5}\lambda c_2$$ ——— (9)

$$\Rightarrow 0c_1 + \left(1 - \frac{2\lambda}{5}\right)c_2 = 0$$

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The characteristic polynomial in λ of the system (8) and (9) is given by

$$D(\lambda) = \begin{vmatrix} 1-\frac{2\lambda}{3} & 0 \\ 0 & 1-\frac{2\lambda}{5} \end{vmatrix} = \left(1-\frac{2\lambda}{3}\right)\left(1-\frac{2\lambda}{5}\right)$$

To find eigen values of eqn (2), we have

$$D(\lambda) = 0 \Rightarrow 1-\frac{2\lambda}{3} = 0 \Rightarrow \lambda = \frac{3}{2} = \lambda_1, \text{ say}$$

$$\text{and } 1-\frac{2\lambda}{5} = 0 \Rightarrow \lambda = \frac{5}{2} = \lambda_2, \text{ say}$$

Case I.

To find eigen function corresponding to eigenvalue $\lambda_1 = \frac{3}{2}$ of the system (8) & (9) $\Rightarrow$

$$\left(1-\frac{2\lambda_1}{3}\right)c_1 + 0c_2 = 0 \Rightarrow \left(1-\frac{2}{3}\cdot\frac{3}{2}\right)c_1 = 0 \Rightarrow 0c_1 = 0$$

$$0c_1 + \left(1-\frac{2\lambda_1}{5}\right)c_2 = 0 \Rightarrow \left(1-\frac{2}{5}\times\frac{3}{2}\right)c_2 = 0$$

$$\Rightarrow \left(1-\frac{3}{5}\right)c_2 = 0$$

$$\Rightarrow c_2 = 0$$

$\therefore$ c_1 is arbitrary

putting $c_2 = 0$ into (4), we have

$$\phi_1(x) = \lambda_1 x c_1 = \frac{3}{2} x c_1 \qquad \left[\because \lambda_1 = \frac{3}{2}\right]$$

$$\Rightarrow \phi_1(x) = \left(\frac{3}{2}c_1\right)x$$

$$\Rightarrow \phi_1(x) = x \qquad \text{if we take } \frac{3}{2}c_1 = 1$$

which is required eigen fn ——— (10)

Now, $\bar{\Phi}_1(x) = \dfrac{\phi_1(x)}{\|\phi_1(x)\|} = \dfrac{x}{\|x\|}$, $\quad \|x\| = \sqrt{\int_{-1}^{1} x^2\,dx}$

$$= \sqrt{\frac{2}{3}}$$

$$\therefore \bar{\Phi}_1(x) = \sqrt{\frac{3}{2}}\cdot x \qquad \text{——— (11)}$$

$$F_1 = \int_{-1}^{1} f(x)\,\bar{\Phi}_1(x)\,dx = \int_{-1}^{1}(x+1)^2\cdot\frac{\sqrt{3}}{\sqrt{2}}\,x\,dx$$

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$$F_1 = \int_{-1}^{1} \frac{\sqrt{3}}{\sqrt{2}}(x^2+2x+1)\, x\, dx = \int_{-1}^{1} \sqrt{\tfrac{3}{2}}(x^3+2x^2+x)\,dx$$

$$F_1 = \sqrt{\tfrac{3}{2}}\left[0+\tfrac{4}{3}+0\right] = \frac{4\sqrt{3}}{\sqrt{2}} = \frac{2\cdot\sqrt{2}\sqrt{2}\sqrt{3}}{3\sqrt{2}}$$

$$\Rightarrow F_1 = \tfrac{2}{3}\sqrt{6} \qquad \text{—— (10)}$$

Case II To find eigen function corresponding to $\lambda_2 = \frac{5}{2}$ of the system (8) & (9), we have

$$\left(1-\tfrac{2}{3}\lambda_2\right)c_1 + 0c_2 = 0 \Rightarrow \left(1-\tfrac{2}{3}\times\tfrac{5}{2}\right)c_1 = 0 \Rightarrow \left(1-\tfrac{5}{3}\right)c_1 = 0$$
$$\Rightarrow c_1 = 0$$

$$0c_1 + \left(1-\tfrac{2}{5}\lambda_2\right)c_2 = 0 \Rightarrow \left(1-\tfrac{2}{5}\times\tfrac{5}{2}\right)c_2 = 0 \Rightarrow 0c_2 = 0$$
$$\Rightarrow c_2 \text{ is arbitrary}$$

so put $c_1 = 0$ in eqn (4), we have

$$\phi_2(x) = \lambda x c_1 + \lambda_2 x^2 c_2$$
$$= 0 + \lambda_2 x^2 c_2$$

$$\phi_2(x) = \left(\tfrac{5}{2}c_2\right)x^2$$

$\Rightarrow \phi_2(x) = x^2$ if we take $\frac{5}{2}c_2 = 1$.

when is required eigen function.

Now,

$$\Phi_2(x) = \frac{\phi_2(x)}{\|\phi_2(x)\|} = \frac{x^2}{\|x^2\|}, \quad \|x^2\| = \sqrt{\int_{-1}^{1} x^4\,dx}$$
$$\therefore \|x^2\| = \sqrt{\tfrac{2}{5}}$$

$$\text{i.e. } \Phi_2(x) = \frac{\sqrt{5}\,x^2}{\sqrt{2}} \qquad \text{—— (11)}$$

Now,
$$F_2 = \int_{-1}^{1} f(x)\,\Phi_2(x)\,dx$$
$$= \int_{-1}^{1}(x+1)^2 \cdot \frac{\sqrt{5}}{\sqrt{2}} \cdot x^2\,dx = \int_{-1}^{1} \frac{\sqrt{5}}{\sqrt{2}}(x^4+2x^3+x^2)\,dx$$
$$= \sqrt{\tfrac{5}{2}}\left[\tfrac{2}{5}+0+\tfrac{2}{3}\right] = \frac{\sqrt{5}}{\cancel{\sqrt{2}}} \times \frac{16}{15} \times \frac{\cancel{\sqrt{2}}}{\sqrt{2}} = \frac{\sqrt{5}\sqrt{2}\times 16}{2\times 15}$$

$$F_2 = \tfrac{8}{15}\sqrt{10} \qquad \text{—— (12)}$$

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11 Required solⁿ of the given equation is

$$\phi(x) = (x+1)^2 + \frac{\lambda}{\lambda_1-\lambda} F_1 \Phi_1(x) + \frac{\lambda}{\lambda_2-\lambda} F_2 \Phi_2(x)$$

$$\phi(x) = (x+1)^2 + \frac{1}{\frac{3}{2}-1}\left(\frac{2}{3}\sqrt{6}\right)\frac{\sqrt{3}}{\sqrt{2}}x + \frac{1}{\left(\frac{5}{2}-1\right)}\frac{8}{15}\sqrt{10}\cdot\frac{\sqrt{5}}{\sqrt{2}}x^2$$

$$\phi(x) = (x+1)^2 + \frac{4}{3}\sqrt{6}\times\frac{\sqrt{3}}{\sqrt{2}}x + \frac{2}{3}\times\frac{8}{15}\times\sqrt{10}\times\frac{\sqrt{5}}{\sqrt{2}}x^2$$

$$= (x+1)^2 + 4x + \frac{16}{9}x^2$$

$$= x^2 + 2x + 1 + 4x + \frac{16}{9}x^2$$

$$\boxed{\phi(x) = \frac{25}{9}x^2 + 6x + 1}$$

(3) Using Hilbert-schmidt theorem, solve

$$\phi(x) = (x^2+1) + \frac{3}{2}\int_{-1}^{1}(x\xi + x^2\xi^2)\phi(\xi)\,d\xi$$

Solution. Given

$$\phi(x) = (x^2+1) + \frac{3}{2}\int_{-1}^{1}(x\xi + x^2\xi^2)\phi(\xi)\,d\xi \qquad (1)$$

Here $\lambda = \frac{3}{2}$.

The homogeneous equation of (1) is

$$\phi(x) = \lambda\int_{-1}^{1}(x\xi + x^2\xi^2)\phi(\xi)\,d\xi \qquad (2)$$

$$\Rightarrow \phi(x) = \lambda x c_1 + \lambda x^2 c_2 \qquad (3)$$

where

$$c_1 = \int_{-1}^{1}\xi\phi(\xi)\,d\xi \qquad (4)$$

$$c_2 = \int_{-1}^{1}\xi^2\phi(\xi)\,d\xi \qquad (5)$$

from (3),

$$\phi(\xi) = \lambda c_1\xi + \lambda c_2\xi^2 \qquad (6)$$

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Equ (4) and (6) $\Rightarrow$

$$c_1 = \int_{-1}^{1} \xi[\lambda c_1 \xi + \lambda c_2 \xi^2]\, d\xi$$

$$c_1 = \lambda c_1 \left(\frac{2}{3}\right) + \lambda c_2 \cdot 0$$

$$\Rightarrow \left(1 - \frac{2\lambda}{3}\right) c_1 + 0 c_2 = 0 \qquad \text{—— (7)}$$

Equ (5) and (6) $\Rightarrow$

$$c_2 = \int_{-1}^{1} \xi^2 [\lambda c_1 \xi + \lambda c_2 \xi^2]\, d\xi$$

$$c_2 = \lambda c_1 [0] + \frac{2}{5}\lambda c_2$$

$$\Rightarrow 0 c_1 + \left(1 - \frac{2\lambda}{5}\right) c_2 = 0 \qquad \text{—— (8)}$$

The characteristic polynomial of (7) & (8) is given by

$$D(\lambda) = \begin{vmatrix} 1 - \frac{2\lambda}{3} & 0 \\ 0 & 1 - \frac{2\lambda}{5} \end{vmatrix}$$

To find eigen values for (2), put $D(\lambda) = 0$

$$\Rightarrow \left(1 - \frac{2\lambda}{3}\right)\left(1 - \frac{2\lambda}{5}\right) = 0 \Rightarrow \lambda = \frac{3}{2} = \lambda_1, \text{ say}$$

and $\lambda = \frac{5}{2} = \lambda_2$, say

Case I To find eigen function corresponding to eigen value $\lambda_1 = \frac{3}{2}$

system (7) and (8) $\Rightarrow$

$$\left(1 - \frac{2\lambda_1}{3}\right) c_1 = 0 \Rightarrow \left(1 - \frac{2}{3} \times \frac{3}{2}\right) c_1 = 0$$

$$\Rightarrow 0 c_1 = 0$$

$\Rightarrow c_1$ is arbitrary

$$\left(1 - \frac{2}{5}\lambda_1\right) c_2 = 0 \Rightarrow \left(1 - \frac{2}{5} \times \frac{3}{2}\right) c_2 = 0$$

$$\Rightarrow c_2 = 0$$

put $c_2 = 0$ into equ (3), we have

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$$\phi(x) = \lambda_1 x c_1 + \lambda_1 x^2 c_2$$

$$\phi(x) = \lambda_1 x c_1$$

$\Rightarrow \phi_1(x) = x$ if we take $\lambda_1 c_1 = 1$, $\lambda_1 = \frac{3}{2}$

which is required eigen function corresponding to eigen value $\lambda_1 = \frac{3}{2}$

Now,

$$\Phi_1(x) = \frac{x}{\|x\|}, \quad \|x\| = \sqrt{\int_{-1}^{1} x^2 dx} = \sqrt{\frac{2}{3}}$$

i $\Phi_1(x) = \sqrt{\frac{3}{2}} \cdot x$ ——— (9)

$$F_1 = \int_{-1}^{1} f(x)\, \Phi_1(x)\, dx$$

$$= \int_{-1}^{1} (x^2+1) \sqrt{\frac{3}{2}} x \, dx$$

$$F_1 = \sqrt{\frac{3}{2}} \int_{-1}^{1} (x^3 + x)\, dx = \sqrt{\frac{3}{2}} [0 + 0]$$

$\Rightarrow F_1 = 0$

Case II To find eigen function corresponding to eigen value $\lambda_2 = \frac{5}{2}$.

system (7) and (8) $\Rightarrow$

$$\left(1 - \frac{2\lambda_2}{3}\right) c_1 = 0 \Rightarrow \left(1 - \frac{2}{3} \cdot \frac{5}{2}\right) c_1 = 0 \Rightarrow c_1 = 0$$

$$\left(1 - \frac{2}{5}\lambda_2\right) c_2 = 0 \Rightarrow \left(1 - \frac{2}{5} \times \frac{5}{2}\right) c_2 = 0 \Rightarrow 0 c_2 = 0$$

$\Rightarrow c_2$ is arbitrary

put $c_1 = 0$ into equation (3), we have

$$\phi(x) = \lambda_2 x c_1 + \lambda_2 x^2 c_2$$

$$\phi(x) = \lambda_2 x \times 0 + \lambda_2 x^2 c_2$$

$$\Rightarrow \phi(x) = \lambda_2 c_2 x^2$$

$\Rightarrow \phi_2(x) = x^2$, if we take $\lambda_2 c_2 = 1$, $\lambda_2 = \frac{5}{2}$.

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$$\Phi_2(x) = \frac{\phi_2(x)}{\|\phi_2(x)\|} = \frac{x^2}{\|x^2\|}, \quad \|x^2\| = \sqrt{\int_{-1}^{1} x^4 dx}$$

$$\therefore \Phi_2(x) = \frac{x^2 \times \sqrt{5}}{\sqrt{2}} \qquad = \sqrt{\frac{2}{5}}$$

$$F_2 = \int_{-1}^{1} f(x)\,\Phi_2(x)\,dx$$

$$= \int_{-1}^{1} (x^2+1)\cdot\sqrt{\frac{5}{2}}\cdot x^2\,dx$$

$$= \sqrt{\frac{5}{2}} \int_{-1}^{1} (x^4+x^2)\,dx = \sqrt{\frac{5}{2}}\left[\frac{2}{5}+\frac{2}{3}\right]$$

$$F_2 = \frac{16}{15} \times \sqrt{\frac{5}{2}}$$

$\therefore$ Required solution of the eqn (1) is given by

$$\phi(x) = f(x) + c\,\Phi_1(x) + \frac{\lambda}{\lambda_2-\lambda}\cdot F_2\,\Phi_2(x)$$

$$(\lambda = \lambda_1 \;\&\; F_1 = 0)$$

$$\phi(x) = (x^2+1) + c\cdot\frac{\sqrt{3}}{\sqrt{2}}x + \frac{\frac{3}{2}}{\frac{5}{2}-\frac{3}{2}}\cdot\frac{16}{15}\frac{\sqrt{5}}{\sqrt{2}} \times \frac{\sqrt{5}}{\sqrt{2}}x^2$$

$$\phi(x) = (x^2+1) + \frac{c\sqrt{6}}{2}x + \frac{\frac{3}{2}}{\frac{2}{2}} \times \frac{16}{15} \times \frac{5}{2}x^2$$

$$\phi(x) = (x^2+1) + \frac{c}{2}\sqrt{6}\,x + \frac{3}{2} \times \frac{16 \times 5}{15 \times 2}x^2$$

$$\phi(x) = x^2+1+\frac{c}{2}\sqrt{6}\,x + 4x^2$$

$$\phi(x) = 5x^2 + Ax + 1, \quad A = \frac{c}{2}\sqrt{6}$$

which is required soln of (1)

Since c is arbitrary $\therefore$ eqn (1) has infinitely many solutions.

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