SUBJECT: MATHEMATICS

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For practice

Q. Find $R(x,\xi;\lambda)$ for $K(x,\xi) = -2+3(x-\xi)$, $\lambda=1$

Ans. $R(x,\xi;1) = \frac{1}{4}e^{x-\xi} - \frac{9}{4}e^{-3(x-\xi)}$.

Iterated kernels

The nth iterated kernel $K_n(x,\xi)$ is determined by the relation

$$K_n(x,\xi) = \int_a^b K_m(x,z)K_{n-m}(z,\xi)\,dz, \quad m<n$$

We know that $K_1(x,\xi) = K(x,\xi)$

i.e. $K_2(x,\xi) = \int_a^b K_1(x,z)K_1(z,\xi)\,dz$

$K_3(x,\xi) = \int_a^b K_1(x,z)K_2(z,\xi)\,dz$ etc

Also we know that

$$R(x,\xi;\lambda) = K(x,\xi) + \lambda K_2(x,\xi) + \lambda^2 K_3(x,\xi) + \cdots$$

Example

(1) Find the iterated kernel of $K(x,\xi) = x-\xi$, $a=0, b=1$.

Solution. Given $K(x,\xi) = x-\xi$, $a=0$, $b=1$

To find iterated kernels (i.e. $K_2(x,\xi), K_3(x,\xi), \cdots$)

We know that $K(x,\xi) = K_1(x,\xi) = (x-\xi)$

i.e. $K_2(x,\xi) = \int_a^b K_1(x,z)K_1(z,\xi)\,dz$

$$= \int_a^b (x-z)(z-\xi)\,dz$$

$$= \int_0^1 (xz - x\xi - z^2 + z\xi)\,dz$$

$$K_2(x,\xi) = \frac{x}{2} - x\xi - \frac{1}{3} + \frac{\xi}{2} = \left(\frac{x+\xi}{2} - x\xi - \frac{1}{3}\right)$$

$$K_3(x,\xi) = \int_0^1 K_1(x,z)\,K_2(z,\xi)\,dz$$

$$= \int_0^1 (x-z)\cdot\left(\frac{z+\xi}{2} - z\xi - \frac{1}{3}\right) dz$$

$$= \int_0^1 \left[\frac{x(z+\xi)}{2} - xz\xi - \frac{x}{3} - z\frac{(z+\xi)}{2} + z^2\xi + \frac{z}{3}\right] dz$$

$$= \int_0^1 \left[\frac{1}{2}(xz + x\xi) - xz\xi - \frac{x}{3} - \frac{1}{2}(z^2 + z\xi) + z^2\xi + \frac{z}{3}\right] dz$$

$$= \left[\frac{x}{4} + \underline{\underline{\frac{x\xi}{2}}} - \underline{\underline{\frac{x\xi}{2}}} - \frac{x}{3} - \cancel{\frac{1}{6}} - \frac{1}{2}\cdot\frac{1}{2}\xi + \frac{\xi}{3} + \cancel{\frac{1}{6}}\right]$$

$$K_3(x,\xi) = \left[\frac{-x}{12} + \frac{\xi}{12}\right] = (-1)\left(\frac{x-\xi}{12}\right)$$

$$K_4(x,\xi) = \int_0^1 K_1(x,z)\,K_3(z,\xi)\,dz$$

$$= \int_0^1 (x-z)(-)\frac{(z-\xi)}{12}\,dz$$

$$= -\frac{1}{12}\int_0^1 (xz - x\xi - z^2 + z\xi)\,dz$$

$$K_4(x,\xi) = -\frac{1}{12}\left[\frac{x}{2} - x\xi - \frac{1}{3} + \frac{\xi}{2}\right] = -\frac{1}{12}\left[\frac{x+\xi}{2} - x\xi - \frac{1}{3}\right]$$

$$K_5(x,\xi) = \int_0^1 K_1(x,z)\,K_4(z,\xi)\,dz$$

$$= \int_0^1 (x-z)\left(-\frac{1}{12}\right)\left(\frac{+\xi}{2} - \xi - \frac{1}{3}\right) dz$$

$$= \left(-\frac{1}{12}\right)\int_0^1 (x-z)\left(\frac{z+\xi}{2} - z\xi - \frac{1}{3}\right) dz$$

$$= \left(-\frac{1}{12}\right)\left(-\frac{1}{12}\right)(x-\xi) \quad \text{as in } K_3(x,\xi)$$

$$K_5(x,\xi) = (-1)^2 \cdot \frac{1}{12^2}(x-\xi)$$

--- and so on.

(2) Find the iterated kernel for $K(x,\xi) = \sin(x-2\xi)$, $0 \le x \le 2\pi$, $0 \le \xi \le 2\pi$.

Solution. Given $K(x,\xi) = \sin(x-2\xi)$, $0 \le x \le 2\pi$, $0 \le \xi \le 2\pi$

We know that $K_1(x,\xi) = K(x,\xi) = \sin(x-2\xi)$

$$K_2(x,\xi) = \int_0^{2\pi} K_1(x,z) K_1(z,\xi)\, dz = \int_0^{2\pi} \sin(x-2z)\sin(z-2\xi)\, dz$$

$$= \frac{1}{2}\int_0^{2\pi} 2\sin(x-2z)\sin(z-2\xi)\, dz$$

$$= \frac{1}{2}\int_0^{2\pi} [\cos(x+2\xi-3z) - \cos(x-2\xi-z)]\, dz$$

$$= \frac{1}{2}\left[\frac{\sin(x+2\xi-3z)}{-3} - \frac{\sin(x-2\xi-z)}{-1}\right]_0^{2\pi}$$

$$= -\frac{1}{6}[\sin(x+2\xi-6\pi) - \sin(x+2\xi)]$$
$$+ \frac{1}{2}[\sin(x-2\xi-2\pi) - \sin(x-2\xi)]$$

$$= -\frac{1}{6}[-\sin(6\pi-\{x+2\xi\}) - \sin(x+2\xi)]$$
$$+ \frac{1}{2}[-\sin\{2\pi-(x-2\xi)\} - \sin(x-2\xi)]$$

$$= -\frac{1}{6}[\cancel{\sin(x+2\xi)} - \cancel{\sin(x+2\xi)}]$$
$$+ \frac{1}{2}[\underline{\sin(x-2\xi)} - \underline{\sin(x-2\xi)}] = 0.$$

$$K_3(x,\xi) = \int_0^{2\pi} K_1(x,z) K_2(z,\xi)\, dz = \int_0^{2\pi} K_1(x,z)\cdot 0\, dz = 0$$

Similarly

$K_4(x,\xi) = 0 = K_5(x,\xi) = \cdots$.

$\therefore R(x,\xi;\lambda) = K_1(x,\xi) + \lambda K_2(x,\xi) + \lambda^2 K_3(x,\xi) + \cdots$

values of. $K_1(x,\xi), K_2(x,\xi), \cdots$ etc

$R(x,\xi;\lambda) = \sin(x-2\xi) + 0 + 0.$

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(3) Find the iterated kernel of $K(x,\xi) = x + \sin\xi$, $a = -\pi$, $b = \pi$

Solution. Given $K_1(x,\xi) = K(x,\xi) = x + \sin\xi$, $a = -\pi$, $b = \pi$

To find iterated kernels

$$K_2(x,\xi) = \int_{-\pi}^{\pi} K_1(x,z)\, K_1(z,\xi)\, dz$$

$$K_2(x,\xi) = \int_{-\pi}^{\pi} (x + \sin z)(z + \sin\xi)\, dz$$

$$K_2(x,\xi) = \int_{-\pi}^{\pi} xz\, dz + x\sin\xi \cdot \int_{-\pi}^{\pi} dz + \int_{-\pi}^{\pi} z \cdot \sin z\, dz + \sin\xi \cdot \int_{-\pi}^{\pi} \sin z\, dz$$

$$K_2(x,\xi) = \frac{x}{2}(z^2)_{-\pi}^{\pi} + x\sin\xi\,[z]_{-\pi}^{\pi} + \left[-z\cos z\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\cos z\, dz\right]$$

$$- \sin\xi\,[\cos z]_{-\pi}^{\pi}$$

$$K_2(x,\xi) = 0 + x\sin\xi\,[2\pi] - [\pi\cos\pi - (-\pi)\cos(-\pi)] + 0 + 0$$

$$K_2(x,\xi) = 2\pi x\sin\xi - [-\pi + \pi\cos\pi]$$

$$K_2(x,\xi) = 2\pi + 2\pi x\sin\xi \Rightarrow K_2(x,\xi) = 2\pi[1 + x\sin\xi]$$

$$K_3(x,\xi) = \int_{-\pi}^{\pi} K_1(x,z)\, K_2(z,\xi)\, dz$$

$$= \int_{-\pi}^{\pi} (x + \sin z) \cdot 2\pi[1 + z\sin\xi]\, dz$$

$$= 2\pi\int_{-\pi}^{\pi} [x + xz\sin\xi + \sin z + \sin\xi \cdot (z\sin z)]\, dz$$

$$= 2\pi\left[x(z)_{-\pi}^{\pi} + \frac{x}{2}\sin\xi\,[z^2]_{-\pi}^{\pi} - [\cos z]_{-\pi}^{\pi} + \sin\xi\left\{-z\cos z\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\cos z\, dz\right\}\right]$$

$$= 2\pi\left[x \cdot 2\pi + 0 - 0 + \sin\xi\left\{-\pi\cos\pi + (-\pi)\cos(-\pi)\right\} + 0\right]$$

$$K_3(x,\xi) = (2\pi)^2 x + 2\pi\sin\xi\,[\pi + \pi] = (2\pi)^2[x + \sin\xi]$$

Similarly

$$K_4(x,\xi) = (2\pi)^3[1 + x\sin\xi],\quad K_5(x,\xi) = (2\pi)^4[x + \sin\xi] \text{ etc.}$$

(4) Find the iterated kernels of $K(x,\xi) = e^{x}\cos\xi$, $a=0$, $b=\pi$

Solution. Given $K_1(x,\xi) = K(x,\xi) = e^{x}\cos\xi$, $a=0$, $b=\pi$

$$K_2(x,\xi) = \int_0^{\pi} K_1(x,z)K_1(z,\xi)\,dz = \int_0^{\pi} e^{x}\cos z \cdot e^{z} \cdot \cos\xi\,dz$$

$$K_2(x,\xi) = e^{x}\cos\xi \cdot \int_0^{\pi} e^{z}\cos z\,dz$$

$$K_2(x,\xi) = e^{x}\cos\xi\left[\frac{e^{z}}{2}\left[\cos z + \sin z\right]\right]_0^{\pi}$$

$$K_2(x,\xi) = \frac{e^{x}\cos\xi}{2}\left[e^{\pi}(\cos\pi + \sin\pi) - (\cos 0 + \sin 0)\right]$$

$$K_2(x,\xi) = \frac{e^{x}\cos\xi}{2}\left[-e^{\pi} + 0 - 1 - 0\right] = -\frac{e^{x}\cos\xi}{2}(1+e^{\pi})$$

$$K_3(x,\xi) = \int_0^{\pi} K_1(x,z)K_2(z,\xi)\,dz$$

$$= \int_0^{\pi} e^{x}\cos z \cdot (-)\frac{e^{z}\cos\xi}{2}(1+e^{\pi})\,dz$$

$$= \frac{(-)(1+e^{\pi})e^{x}\cos\xi}{2}\int_0^{\pi} e^{z}\cos z\,dz =$$

$$K_3(x,\xi) = \frac{(-1)^2}{2^2} e^{x}\cos\xi\,(1+e^{\pi})^2$$

Similarly

$$K_4(x,\xi) = \frac{(-1)^3}{2^3}(1+e^{\pi})^3 \cdot e^{x}\cos\xi$$

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(5) Find the resolvent kernel of $K(x,\xi) = (1+x)(1-\xi)$, $a=-1$, $b=0$

Solution. Given $K_1(x,\xi) = (1+x)(1-\xi)$, $a=-1$, $b=0$.

[$\because K(x,\xi) = K_1(x,\xi)$]

$$K_2(x,\xi) = \int_{-1}^{0} K_1(x,z)K_1(z,\xi)\,dz$$

$$K_2(x,\xi) = \int_{-1}^{0}(1+x)(1-z)(1+z)(1-\xi)\,dz$$

$$K_2(x,\xi) = \int_{-1}^{0} (1+x)(1-\xi).(1-z^2)\,dz$$

$$K_2(x,\xi) = (1+x)(1-\xi)\left[z - \frac{z^3}{3}\right]_{-1}^{0}$$

$$K_2(x,\xi) = (1+x)(1-\xi)\left[(0) - \left\{(-1) - \frac{(-1)^3}{3}\right\}\right]$$

$$K_2(x,\xi) = (1+x)(1-\xi)\left[1 - \frac{1}{3}\right] = \frac{2}{3}(1+x)(1-\xi)$$

$$K_3(x,\xi) = \int_{-1}^{0} K_1(x,z)\,K_2(z,\xi)\,dz$$

$$K_3(x,\xi) = \int_{-1}^{0} (1+x)(1-z) \cdot \frac{2}{3}(1+z)(1-\xi)\,dz$$

$$K_3(x,\xi) = (1+x)(1-\xi) \cdot \frac{2}{3} \int_{-1}^{0} (1-z)(1+z)\,dz$$

$$K_3(x,\xi) = (1+x)(1-\xi) \cdot \frac{2}{3}\left[\frac{2}{3}\right] = \left(\frac{2}{3}\right)^2 (1+x)(1-\xi)$$

Similarly

$$K_4(x,\xi) = \left(\frac{2}{3}\right)^3 (1+x)(1-\xi) \;\text{------}$$

$\therefore R(x,\xi;\lambda) = K_1(x,\xi) + \lambda K_2(x,\xi) + \lambda^2 K_3(x,\xi) + \text{---}$ can be obtained by putting the values.

(6) Find resolvant kernel.

Solution. Given $K_1(x,\xi) = K(x,\xi) = e^{x+\xi}$, $a = 0$, $b = 1$

$$K_2(x,\xi) = \int_0^1 K_1(x,z)\,K_1(z,\xi)\,dz$$

$$= \int_0^1 e^{x+z} \cdot e^{z+\xi}\,dz = e^{x+\xi} \cdot \int_0^1 e^{2z}\,dz$$

$$K_2(x,\xi) = e^{x+\xi}\left(\frac{e^2-1}{2}\right)$$

$$K_3(x,\xi) = \int_0^1 K_1(x,z)\,K_2(z,\xi)\,dz = \int_0^1 e^{x+z} \cdot e^{z+\xi}\left(\frac{e^2-1}{2}\right)dz$$

$$K_3(x,\xi) = \left(\frac{e^2-1}{2}\right) \cdot e^{x+\xi} \cdot \int_0^1 e^{2z}\,dz = \frac{(e^2-1)^2}{2^2}\,e^{x+\xi}$$

Similarly find $K_4(x,\xi)$, $K_5(x,\xi)$ --- & final $R(x,\xi;\lambda)$ as in Q.5.

(7) Solve $\phi(x) = x + \int_0^{\frac{1}{2}} \phi(\xi)\, d\xi$ —— (1)

Sol^n of (1) is given by

$$\phi(x) = x + \int_0^{\frac{1}{2}} R(x,\xi;1)\cdot \xi\, d\xi \quad \text{—— (2)}$$

where

$$R(x,\xi;1) = K_1(x,\xi) + K_2(x,\xi) + K_3(x,\xi) + K_4(x,\xi) + \cdots \quad \text{—— (3)}$$

Here

$K_1(x,\xi) = 1$

$K_2(x,\xi) = \int_0^{\frac{1}{2}} K_1(x,z)\, K_1(z,\xi)\, dz = \int_0^{\frac{1}{2}} 1.1\, dz = \frac{1}{2}$

$K_3(x,\xi) = \int_0^{\frac{1}{2}} K_1(x,z)\, K_2(z,\xi)\, dx = \int_0^{\frac{1}{2}} 1.\ \frac{1}{2}\, dz = \left(\frac{1}{2}\right)^2$

Similarly

$K_4(x,\xi) = \left(\frac{1}{3}\right)^3, \cdots$

$\therefore\ R(x,\xi;1) = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots$

$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = \frac{1}{1-\frac{1}{2}} = 2$

$\therefore$ Eq^ns (2) $\Rightarrow$

$\phi(x) = x + \int_0^{\frac{1}{2}} 2.\xi\, dz = x + (\xi^2)_0^{\frac{1}{2}} = x + \frac{1}{4}.$

which is required Solution.

(8) Solve $\phi(x) = \left(\sin x - \frac{x}{4}\right) + \frac{1}{4}\int_0^{\frac{\pi}{2}} \xi x\, \phi(\xi)\, d\xi$

Solution. Given

$$\phi(x) = \left(\sin x - \frac{x}{4}\right) + \frac{1}{4}\int_0^{\frac{\pi}{2}} \xi x\, \phi(\xi)\, d\xi \quad \text{—— (1)}$$

Its Solution is given by

$$\phi(x) = \left(\sin x - \frac{x}{4}\right) + \frac{1}{4}\int_0^{\frac{\pi}{2}} R(x,\xi;\tfrac{1}{4})\cdot\left(\sin\xi - \frac{\xi}{4}\right) d\xi \quad \text{—— (2)}$$

where

$$R(x,\xi;\tfrac{1}{4}) = K_1(x,\xi) + \frac{1}{4}K_2(x,\xi) + \left(\frac{1}{4}\right)^2 K_3(x,\xi) + \cdots \quad \text{—— (3)}$$

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$$K_1(x,\xi) = K(x,\xi) = x\xi$$

$$K_2(x,\xi) = \int_0^{\frac{\pi}{2}} K_1(x,z)\,K_1(z,\xi)\,dz$$

$$= \int_0^{\frac{\pi}{2}} xz \cdot z\xi\,dz = x\xi \cdot \left(\frac{z^3}{3}\right)_0^{\frac{\pi}{2}}$$

$$K_2(x,\xi) = \frac{x\xi}{3}\cdot\left(\frac{\pi}{2}\right)^3$$

$$K_3(x,\xi) = \int_0^{\frac{\pi}{2}} K_1(x,z)\,K_2(z,\xi)\,dz$$

$$= \int_0^{\frac{\pi}{2}} xz \cdot \frac{z\xi}{3}\left(\frac{\pi}{2}\right)^3 dx = \frac{1}{3}\left(\frac{\pi}{2}\right)^3 \cdot x\xi \int_0^{\frac{\pi}{2}} z^2\,dz$$

$$= \frac{1}{3}\left(\frac{\pi}{2}\right)^3 \cdot x\xi \cdot \frac{1}{3}\cdot\left(\frac{\pi}{2}\right)^3$$

$$K_3(x,\xi) = \left(\frac{1}{3}\right)^2\left(\frac{\pi^3}{2^3}\right)^2$$

— — — and so on

$$R(x,\xi;\tfrac{1}{4}) = x\xi + \frac{1}{4}\,\frac{\pi^3}{2^3}\cdot\frac{1}{3}x\xi + \left(\frac{1}{4}\right)^2\cdot\left(\frac{\pi^3}{2^3}\right)^2\cdot\frac{1}{3^2}\,x\xi + \cdots$$

$$= x\xi\left[1 + \frac{\pi^3}{96} + \left(\frac{\pi^3}{96}\right)^2 + \cdots\right]$$

$$R(x,\xi;\tfrac{1}{4}) = x\xi\left[\frac{1}{1-\frac{\pi^3}{96}}\right] = x\xi\left[\frac{96}{96-\pi^3}\right]$$

$\therefore$ Required Solution is given by

$$\phi(x) = \left(\sin x - \frac{x}{4}\right) + \frac{1}{4}\int_0^{\frac{\pi}{2}} x\xi\left(\frac{96}{96-\pi^3}\right)\left(\sin\xi - \frac{\xi}{4}\right)d\xi$$

$$\phi(x) = \left(\sin x - \frac{x}{4}\right) + \frac{24x}{96-\pi^3}\cdot\int_0^{\pi/2}\left(\xi\sin\xi - \frac{\xi^2}{4}\right)d\xi \quad \text{--- (4)}$$

$$\text{Let } I_1 = \int_0^{\pi/2}\left(\xi\sin\xi - \frac{\xi^2}{4}\right)d\xi \quad \text{--- (5)}$$

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$\therefore$ (4) & (5) $\Rightarrow$

$$\phi(x) = (\sin x - \frac{x}{4}) + \frac{24x}{96-\pi^3} I_1 \quad \text{---} (6)$$

where

$$I_1 = \int_0^{\frac{\pi}{2}} (\xi \sin\xi - \frac{\xi^2}{4}) d\xi$$

$$= -\xi\cos\xi \Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos\xi \, d\xi - \frac{1}{4}\int_0^{\frac{\pi}{2}} \xi^2 d\xi$$

$$= 0 + (\sin\xi)_0^{\frac{\pi}{2}} - \frac{1}{4}\left[\frac{\xi^3}{3}\right]_0^{\pi/2}$$

$$I_1 = 1 - \frac{1}{12}\cdot\frac{\pi^3}{8} = \frac{96-\pi^3}{96} \quad \text{---} (7)$$

$\therefore$ Eqⁿs (6) & (7) $\Rightarrow$

$$\phi(x) = (\sin x - \frac{x}{4}) + \frac{24x}{\cancel{(96-\pi^3)}} \cdot \frac{\cancel{(96-\pi^3)}}{96}$$

$$= \sin x - \frac{x}{4} + \frac{x}{4}$$

$\phi(x) = \sin x$, which is required Solution.

(9) Solve the integral eqⁿs

$$\phi(x) = f(x) + \lambda\int_0^1 e^{x-\xi}\phi(\xi)d\xi$$

Solution. Given $\phi(x) = f(x) + \lambda\int_0^1 e^{x-\xi}\phi(\xi)d\xi$ — (1)

Its solution is

$$\phi(x) = f(x) + \lambda\int_0^1 R(x,\xi;\lambda) f(\xi)d\xi \quad \text{---} (2)$$

where

$$R(x,\xi;\lambda) = K_1(x,\xi) + \lambda K_2(x,\xi) + \lambda^2 K_3(x,\xi) + \cdots \quad (3)$$

Here $K_1(x,\xi) = K(x,\xi) = e^{x-\xi}$

$$K_2(x,\xi) = \int_0^1 K_1(x,z)K_1(z,\xi)dz = \int_0^1 e^{x-z}\cdot e^{z-\xi}dz$$

$$K_2(x,\xi) = e^{x-\xi}\int_0^1 dz = e^{x-\xi}$$

Similarly

$$K_3(x,\xi) = e^{x-\xi}, \quad K_4(x,\xi) = e^{x-\xi}$$

$\therefore R(x,\xi;\lambda) = e^{x-\xi}[1+\lambda+\lambda^2+\lambda^3+\cdots] = e^{x-\xi}\frac{1}{1-\lambda}$

$\therefore$ Req. solⁿ is $\phi = f(x) + \lambda\int_0^1 e^{x-\xi}\frac{1}{1-\lambda}\cdot f(\xi)d\xi$, Ans.

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(10) $\phi(x) = 1+\lambda\int_0^{\pi} \sin(x+\xi)\phi(\xi)\,d\xi$

Solution. Given $\phi(x) = 1+\lambda\int_0^{\pi}\sin(x+\xi)\phi(\xi)\,d\xi$ —— (1)

Its solution is

$$\phi(x) = 1+\lambda\int_0^{\pi} R(x,\xi;\lambda)\cdot 1\,d\xi \quad —— (2)$$

where

$$R(x,\xi;\lambda) = K_1(x,\xi)+\lambda K_2(x,\xi)+\lambda^2 K_3(x,\xi)+ \cdots \quad —— (3)$$

where $\boxed{K_1(x,\xi) = \sin(x+\xi)}$ —— (4)

$$K_2(x,\xi) = \int_0^{\pi} K_1(x,z)K_1(z,\xi)\,dz$$

$$K_2(x,\xi) = \int_0^{\pi}\sin(x+z)\sin(z+\xi)\,dz$$

$$K_2(x,\xi) = \frac{1}{2}\int_0^{\pi} 2\sin(x+z)\sin(z+\xi)\,dz$$

$$K_2(x,\xi) = \frac{1}{2}\int_0^{\pi}[\cos(x-\xi)-\cos(x+\xi+2z)]\,dz$$

$$K_2(x,\xi) = \frac{1}{2}\cos(x-\xi)\int_0^{\pi}dz - \frac{1}{2}\int_0^{\pi}\cos(x+\xi+2z)\,dz$$

$$K_2(x,\xi) = \frac{\pi}{2}\cos(x-\xi)-\frac{1}{2}\left[\frac{-\sin(x+\xi+2z)}{2}\right]_0^{\pi}$$

$$K_2(x,\xi) = \frac{\pi}{2}\cos(x-\xi)-\frac{1}{4}[\sin(x+\xi+2\pi)-\sin(x+\xi)]$$

$$K_2(x,\xi) = \frac{\pi}{2}\cos(x-\xi)-\frac{1}{4}[\sin\cancel{(x+\xi)}-\sin\cancel{(x+\xi)}]$$

$\boxed{K_2(x,\xi) = \frac{\pi}{2}\cos(x-\xi)}$ —— (5)

$$K_3(x,\xi) = \int_0^{\pi}K_1(x,z)K_2(z,\xi)\,dz$$

$$= \int_0^{\pi}\sin(x+z)\,\frac{\pi}{2}\cos(z-\xi)\,dz$$

$$= \frac{\pi}{2}\cdot\frac{1}{2}\int_0^{\pi}2\sin(x+z)\cos(z-\xi)\,dz$$

$$K_3(x,\xi) = \frac{\pi}{4}\int_0^{\pi}[\sin(x+2z-\xi)+\sin(x+\xi)]\,dz$$

$$= \frac{\pi}{4}\left[-\left\{\frac{\cos(x+2z-\xi)}{2}\right\}\Big|_0^{\pi}\right]+\frac{\pi}{4}\sin(x+\xi)\cdot\pi$$

$$K_3(x,\xi) = \frac{\pi}{4} \times 0 + \frac{\pi^2}{4}\sin(x+\xi)$$

$$\Rightarrow K_3(x,\xi) = \left(\frac{\pi}{2}\right)^2 \sin(x+\xi) \qquad \text{——— (6)}$$

Similarly

$$K_4(x,\xi) = \left(\frac{\pi}{2}\right)^3 \cos(x-\xi) \qquad \text{——— (7)}$$

$$K_5(x,\xi) = \left(\frac{\pi}{2}\right)^4 \sin(x+\xi) \qquad \text{——— (8)}$$

Putting the values of $K_1(x,\xi)$, $K_2(x,\xi)$, --- into (3), we have

$$\therefore R(x,\xi;\lambda) = \sin(x+\xi) + \lambda\cdot\frac{\pi}{2}\cos(x-\xi) + \left(\frac{\pi}{2}\right)^2\lambda^2\sin(x+\xi) + \left(\frac{\pi}{2}\right)^3\lambda^3\cos(x-\xi) + \cdots$$

$$= \sin(x+\xi)\left[1 + \frac{\pi^2}{4}\lambda^2 + \left(\frac{\pi}{2}\right)^4\lambda^4 + \cdots\right] + \frac{\lambda\pi}{2}\cos(x-\xi)\left[1 + \frac{\lambda^2\pi^2}{2^2} + \frac{\lambda^4\pi^4}{2^4} + \cdots\right]$$

$$= \sin(x+\xi)\left(\frac{1}{1-\frac{\pi^2\lambda^2}{4}}\right) + \frac{\lambda\pi}{2}\cos(x-\xi)\left(\frac{1}{1-\frac{\pi^2\lambda^2}{4}}\right)$$

$$R(x,\xi;\lambda) = \frac{1}{\left(1-\frac{\pi^2\lambda^2}{4}\right)}\left[\sin(x+\xi) + \frac{\lambda\pi}{2}\cos(x-\xi)\right]$$

$\therefore$ Required soln of (1) is

$$\phi(x) = 1 + \frac{\lambda}{1-\frac{\pi^2\lambda^2}{4}}\int_0^{\pi}\left[\sin(x+\xi) + \frac{\pi\lambda}{2}\cos(x-\xi)\right]d\xi$$

$$\phi(x) = 1 + \frac{4\lambda}{4-\pi^2\lambda^2}\cdot\frac{1}{2}\int_0^{\pi}\left[2\sin(x+\xi) + \pi\lambda\cos(x-\xi)\right]d\xi$$

$$\phi(x) = 1 + \frac{2\lambda}{4-\pi^2\lambda^2}\cdot I_1 \qquad \text{——— (9)}$$

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where

$$I_1 = \int_0^{\pi} [2\sin(x+\xi) + \pi\lambda \cos(x-\xi)]\, d\xi$$

$$= -2[\cos(x+\xi)]_0^{\pi} + \frac{\pi\lambda}{(-)}[\sin(x-\xi)]_0^{\pi}$$

$$= -2[\cos(x+\pi) - \cos x] - \pi\lambda[\sin(x-\pi) - \sin x]$$

$$= -2[-\cos x - \cos x] - \pi\lambda[-\sin(\pi-x) - \sin x]$$

$$= 4\cos x + \pi\lambda \cdot 2\sin x$$

$$I_1 = 2[2\cos x + \pi\lambda \sin x] \quad \text{——} \quad (10)$$

Equ's (9) & (10) ⇒

$$\phi(x) = 1 + \frac{2\lambda}{4-\pi^2\lambda^2} \cdot I_1$$

$$\phi(x) = 1 + \frac{2\lambda}{4-\pi^2\lambda^2} \cdot 2[2\cos x + \pi\lambda \sin x]$$

$$\phi(x) = 1 + \frac{4\lambda}{4-\pi^2\lambda^2}(2\cos x + \pi\lambda \sin x)$$

Q solve by yourself

$$\phi(x) = 1 + \lambda \int_0^1 (1+3x\xi)\phi(\xi)\, d\xi$$

Ans $\phi(x) = \dfrac{[4 + 4\lambda(2-3x)]}{(4-\lambda^2)}$.