SUBJECT: MATHEMATICS

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(1) Integral Equations

There are three types of Integral equation

1. Linear integral equation
2. Non-Linear integral equation
3. Singular integral equation

(1) Linear integral equation: An integral equation is called linear integral equation if there is only linear functions as unknown functions under the integral sign.

ex. $\alpha(x)\cdot\phi(x) = f(x) + \lambda \int_a^b K(x,\xi)\,\phi(\xi)\,d\xi$ is linear as the unknown function $K(x,\xi)\,\phi(\xi)$ is linear

This Linear Integral equation is further has been divided into two types:

1. Volterra integral equation
2. Fredholm integral equation

(1) Volterra integral equation An integral equation is said to be Volterra integral equation if the upper limit of integration equation is variable

ex $$\alpha(x)\,\phi(x) = f(x) + \lambda \int_a^x K(x,\xi)\,\phi(\xi)\,d\xi \quad -(1)$$

Here upper limit is x which is variable.

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(i) If $\alpha = 0$ then equation (1) implies

$$F(x) = \lambda \int_a^x K(x,\xi)\,\phi(\xi)\,d\xi$$

which is called Volterra integral equation of first kind.

(ii) If $\alpha = 1$ then equation (1) implies

$$\phi(x) = f(x) + \lambda \int_a^x K(x,\xi)\,\phi(\xi)\,d\xi$$

which is called volteraa integral equation of Second kind.

(iii) If $\alpha = 1$, $f(x) = 0$ then the equation (1) implies

$$\phi(x) = \lambda \int_a^x K(x,\xi)\,\phi(\xi)\,d\xi$$

which is called homogeneous volteraa integral Equation.

(2) Fredholm integral equation. An integral equation is called Fredholm integral equation if both the limits are constants i.e. domain of integration is fixed.

e.g. An equation

$$\alpha(x)\,\phi(x) = f(x) + \lambda \int_a^b K(x,\xi)\,\phi(\xi)\,d\xi \quad \text{is called} \qquad -(2)$$

Fredholm integral equation.

(i) if $\alpha = 0$ then equation (2) implies

$$F(x) = \lambda \int_a^b K(x,\xi)\,\phi(\xi)\,d\xi$$

which is called Fredholm integral equation of first kind.

(ii) if $\alpha = 1$ then the equation (2) implies

$$\phi(x) = f(x) + \lambda \int_a^b K(x,\xi)\,\phi(\xi)\,d\xi$$

which is called Fredholm integral equation of second kind.

(iii) if $\alpha = 1$, $f(x) = 0$ then the equation (2) implies

$$\phi(x) = \lambda \int_a^b K(x,\xi)\,\phi(\xi)\,d\xi$$

which is called homogeneous integral equation of second kind.

(3) Singular integral equation. When one or both limits of integration are infinite or the kernel $K(x,\xi)$ becomes infinite at one or more points within the range of integration, is called Singular integral equation.

ex $$\phi(x) = f(x) + \lambda \int_{-\infty}^{\infty} \exp\{-|x-\xi|\}\,\phi(\xi)\,d\xi$$

and $$\phi(x) = f(x) = \int_a^x \frac{1}{(x-\xi)^\alpha}\,\phi(\xi)\,d\xi \quad , 0 < \alpha < 1$$

(4) Non-linear integral equation. If the unknown function appears under an integral sign to some power greater than one, is called non-linear integral equation.

ex $$\phi(x) = F(x) + \lambda \int_a^b K(x,\xi)\,\phi^n(\xi)\,d\xi \quad (n>1).$$

(5) Convolution integration. If the kernel $K(x,\xi)$ of the integral equation is of the form

$$K(x,\xi) = K(x-\xi)$$

then the integral equation is said to be of convolution type.

ex $\phi(x) = e^{x} + \lambda \int_a^{b} [(x-\xi)^2 + 3(x-\xi)]\phi(\xi)\,d\xi$

<u>Formula</u>: Differentiation of a definite integral

$$\frac{\partial}{\partial x}\left[\int_{P(x)}^{Q(x)} [F(x,\xi)\phi(\xi)]\,d\xi\right.$$

$$= \int_{P(x)}^{Q(x)} \left[\frac{\partial}{\partial x}[F(x,\xi)\phi(\xi)]\right] d\xi + [F(x,\xi)\phi(\xi)]_{\xi=Q(x)} \frac{d}{dx}Q(x)$$

$$- [F(x,\xi)\phi(\xi)]_{\xi=P(x)} \frac{d}{dx}P(x).$$

<u>Examples</u>

(1) Given $\phi(x) = \left(1-x+\frac{x^3}{6}\right) + \int_0^x [\sin\xi - (x-\xi)(\cos\xi + e^{\xi})]\phi(\xi)\,d\xi$ ——— (1)

To find $\phi'(x)$ and $\phi''(x)$.

Differentating (1) with respect to x, we have

$$\phi'(x) = \left(0-1+\frac{3x^2}{6}\right) + \int_0^x \left[\frac{\partial}{\partial x}[\sin\xi - (x-\xi)(\cos\xi + e^{\xi})]\phi(\xi)\right] d\xi$$

$$+ \left[[\sin\xi - (x-\xi)(\cos\xi + e^{\xi})]\phi(\xi)\right]_{\xi=x} \frac{d}{dx}(x)$$

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$$-\left[\left\{\sin\xi-(x-\xi)(\cos\xi+e^{\xi})\right\}\phi(\xi)\right]_{\xi=0}\frac{d}{dx}(0).$$

$$\phi'(x)=-1+\frac{x^2}{2}+\int_0^x\left[0-(\cos\xi+e^{\xi})\right]\phi(\xi)\,d\xi$$

$$+\left[(-\sin x-(x-x)(\cos x+e^{x})\right]\phi(x)\cdot 1-0. \quad \left[\because \frac{d}{dx}0=0\right]$$

$$\boxed{\phi'(x)=-1+\frac{x^2}{2}-\int_0^x(e^{\xi}+\cos\xi)\phi(\xi)\,d\xi+\sin x\,\phi(x)}$$

Again, differentiating, we have

$$\phi''(x)=-0+\frac{2x}{2}-\left[\int_0^x\left\{\frac{\partial}{\partial x}\left[(e^{\xi}+\cos\xi)\phi(\xi)\right]\right\}d\xi\right.$$

$$\left.+\left\{(e^{\xi}+\cos\xi)\phi(\xi)\right\}_{\xi=x}\frac{d}{dx}x-\left[(e^{\xi}+\cos\xi)\phi(\xi)\right]_{\xi=0}\frac{d}{dx}0\right]$$

$$+\sin x\,\phi'(x)+\phi(x)\cos x$$

$$\phi''(x)=x-[0]-(e^{x}+\cos x)\phi(x)+\sin x\cdot\phi'(x)$$
$$+\phi(x)\cos x.$$

$$\phi''(x)=x-e^{x}\phi(x)-\cancel{\cos x\,\phi(x)}+\sin x\cdot\phi'(x)$$
$$+\cancel{\phi(x)\cos x}$$

$$\boxed{\phi''(x)-\sin x\,\phi'(x)+e^{x}\phi(x)=x}$$

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Given

(2) $$\phi(x) = (1 - x - 4\sin x) + \int_0^x [3 - 2(x-\xi)]\phi(\xi)\,d\xi \quad \text{—— (1)}$$

Find $\phi'(x)$ and $\phi''(x)$

Differentiating (1), we have

$$\phi'(x) = (0 - 1 - 4\cos x) + \int_0^x \left[\frac{\partial}{\partial x}\left[(3 - 2(x-\xi))\phi(\xi)\right]\right]d\xi$$

$$+ \left[[3 - 2(x-\xi)]\phi(\xi)\right]_{\xi = x}\frac{d}{dx}x - \left[\{3 - 2(x-\xi)\}\phi(\xi)\right]_{\xi=0}\frac{d}{dx}0.$$

$$\phi'(x) = -1 - 4\cos x + \int_0^x -2\,\phi(\xi)\,d\xi + \left[\{3 - 2(x-x)\}\phi(x)\right]\cdot 1 - 0.$$

$$\phi'(x) = -1 - 4\cos x - 2\int_0^x \phi(\xi)\,d\xi + 3\phi(x)$$

Again, differentiating, we get

$$\phi''(x) = 0 + 4\sin x - 2\left[\int_0^x \left[\frac{\partial}{\partial x}\phi(\xi)\right]d\xi + [\phi(\xi)]_{\xi=x}\frac{d}{dx}x - [\phi(\xi)]_{\xi=0}\frac{d}{dx}0\right] + 3\phi'(x)$$

$$\phi''(x) = 4\sin x - 2\left[0 + \phi(x) - 0\right] + 3\phi'(x)$$

$$\phi''(x) - 3\phi'(x) + 2\phi(x) = 4\sin x$$

(3) If $\phi(x) = 3 + \int_0^x (5x - 3\xi)\phi(\xi)\,d\xi$ —— (1), find $\phi'(x)$, $\phi''(x)$

Differentiating (1) with respect to x, we get

$$\phi'(x) = 0 + \int_0^x \frac{\partial}{\partial x}\left[\{5x - 3\xi\}\phi(\xi)\right]d\xi$$

$$+ \left[(5x - 3\xi)\phi(\xi)\right]_{\xi = x}\frac{d}{dx}x - \left[(5x - 3\xi)\phi(\xi)\right]_{\xi=0}\frac{d}{dx}0$$

$$\phi'(x) = \int_0^x 5\cdot\phi(\xi)\,d\xi + 2x\phi(x)$$

$$\phi'(x) = \int_0^x 5\phi(\xi)d\xi + 2x\phi(x)$$

Again, Differentiating with respect to x, we have

$$\phi''(x) = \int_0^x \frac{\partial}{\partial x}[5\phi(\xi)]\,d\xi + 5[\phi(\xi)]_{\xi=x}\frac{d}{dx}x$$
$$- 5[\phi(\xi)]_{\xi=0}\frac{d}{dx}\cdot 0 + 2[x\phi'(x) + 1\cdot\phi(x)]$$

$$\phi''(x) = 0 + 5\phi(x) - 0 + 2x\phi'(x) + 2\phi(x)$$

$$\phi''(x) - 2x\phi'(x) - 7\phi(x) = 0$$

(4) If $\phi(x) = \int_0^x (x+\xi)\phi(\xi)\,d\xi$, then find $\phi'(x)$, $\phi''(x)$.

Solution. Given $\phi(x) = \int_0^x (x+\xi)\phi(\xi)\,d\xi$

$$\phi'(x) = \int_0^x \frac{\partial}{\partial x}[(x+\xi)\phi(\xi)]\cdot d\xi$$
$$+ [(x+\xi)\phi(\xi)]_{\xi=x}\frac{d}{dx}x - [(x+\xi)\phi(\xi)]_{\xi=0}\frac{d}{dx}\cdot 0.$$

$$\phi'(x) = \int_0^x \phi(\xi)d\xi + 2x\phi(x) - 0$$

$$\phi''(x) = \int_0^x \frac{\partial}{\partial x}[\phi(\xi)]\,d\xi + 2[x\phi'(x) + \phi(x)]$$

$$\phi''(x) = 0 + 2x\phi'(x) + 2\phi(x)$$

or, $\phi''(x) - 2x\phi'(x) - 2\phi(x) = 0.$

Remember

Let $\frac{d}{dx} I_2 = I_1 = \int_a^x f(x_1)\,dx_1$

$$\Rightarrow \quad I_2 = \int_a^x \int_a^{x_2} f(x_1)\,dx_1\,dx_2 = \int_a^x f(\xi)\,d\xi^2$$

$$I_2 = \int_a^x (x-\xi) f(\xi)\,d\xi$$

In general

$$\int_a^x f(\xi)\,d\xi^n = \int_a^x \frac{(x-\xi)^{n-1}}{\lfloor n-1} f(\xi)\,d\xi$$

For $n=3$

$$\int_a^x f(\xi)\,d\xi^3 = \int_a^x \frac{(x-\xi)^{3-1}}{\lfloor 3-1} f(\xi)\,d\xi$$

For $n=4$

$$\int_a^x f(\xi)\,d\xi^4 = \int_a^x \frac{(x-\xi)^3}{\lfloor 3} f(\xi)\,d\xi$$

for $n=5$

$$\int_a^x f(\xi)\,d\xi^5 = \int_a^x \frac{(x-\xi)^4}{\lfloor 4} \cdot f(\xi)\,d\xi$$

Note. Here

$$\int_a^x f(\xi)\,d\xi^n = \int_a^x \int_a^{x_2} \int_a^{x_3} \cdots \int_a^{x_n} f(x_1)\,dx_1\,dx_2 \cdots dx_{n-1}\,dx_n .$$

To convert a differential equation into an Integral equation (non homogeneous Volterra's Integral equation of second kind) and vice-versa. (2004, 07)

(1) Reduce $\phi''(x) - 3\phi'(x) + 2\phi(x) = 4\sin x$ with the initial conditions $\phi(0) = 1$, $\phi'(0) = -2$ into a non homogeneous Volterra's integral equation of second kind. Conversely, derive the original differential equation with the initial conditions from the integral equation obtained.

Solution. The given differential equation may be written as

$$\phi''(x) = 4\sin x - 2\phi(x) + 3\phi'(x) \quad \text{--- (1)}$$

Integrating both sides with respect to x from $x=0$ to $x=x$, we have

$$\int_0^x \phi''(\xi)\,d\xi = 4\int_0^x \sin\xi\,d\xi - 2\int_0^x \phi(\xi)\,d\xi + 3\int_0^x \phi'(\xi)\,d\xi$$

$$\Rightarrow \phi'(x) - \phi'(0) = -4(\cos\xi)_0^x - 2\int_0^x \phi(\xi)\,d\xi + 3[\phi(x) - \phi(0)]$$

$$\Rightarrow \phi'(x) + 2 = -4(\cos x - 1) - 2\int_0^x \phi(\xi)\,d\xi + 3\phi(x) - 3$$

$$\left[\because \phi'(0) = -2 \;\&\; \phi(0) = 1\right]$$

$$\Rightarrow \phi'(x) = -2 - 4\cos x + 4 - 2\int_0^x \phi(\xi)\,d\xi + 3\phi(x) - 3$$

$$\Rightarrow \phi'(x) = -1 - 4\cos x + 3\phi(x) - 2\int_0^x \phi(\xi)\,d\xi \quad \text{--- (2)}$$

Again, integrating both sides with respect to x from $x=0$ to $x=x$, we have

$$\int_0^x \phi'(\xi)d\xi = -\int_0^x d\xi - 4\int_0^x \cos\xi\, d\xi + 3\int_0^x \phi(\xi)d\xi - 2\int_0^x \phi(\xi)\, d\xi^2$$

$$\left[\because \int_0^x \left(\int_0^x \phi(\xi)d\xi\right)d\xi = \int_0^x \phi(\xi)\,d\xi^2\right]$$

$$\Rightarrow \phi(x) - \phi(0) = -x - 4\left[-\sin\xi\right]_0^x + 3\int_0^x \phi(\xi)d\xi - 2\int_0^x (x-\xi)\phi(\xi)d\xi$$

$$\left[\because \int_0^x \phi(\xi)\,d\xi^2 = \int_0^x (x-\xi)\phi(\xi)d\xi\right]$$

$$\Rightarrow \phi(x) - 1 = -x - 4\sin x + 3\int_0^x \phi(\xi)d\xi - 2\int_0^x (x-\xi)\phi(\xi)d\xi$$

$$\Rightarrow \phi(x) = (1 - x - 4\sin x) + \int_0^x \{3 - 2(x-\xi)\}\phi(\xi)\,d\xi \quad —(3)$$

which is non-homogeneous Volterra's integral equation of second kind.

Converse. Suppose

$$\phi(x) = (1 - x - 4\sin x) + \int_0^x \{3 - 2(x-\xi)\}\phi(\xi)\,d\xi \quad —(1)$$

Differentiating both sides with respect to x, we get

$$\phi'(x) = (0 - 1 - 4\cos x) + \frac{\partial}{\partial x}\left[\int_0^x \{3 - 2(x-\xi)\}\phi(\xi)d\xi\right]$$

$$\phi'(x) = -1 - 4\cos x + \int_0^x \frac{\partial}{\partial x}\left[\{3 - 2(x-\xi)\}\phi(\xi)\right].\, d\xi$$

$$+ \left[\{3 - 2(x-\xi)\}\phi(\xi)\right]_{\xi = x} \frac{d}{dx}x - \left[\{3 - 2(x-\xi)\}\phi(\xi)\right]\frac{d}{dx}.0.$$

$$\phi'(x) = -1 - 4\cos x - 2\int_0^x \phi(\xi)d\xi + \{3 - 2(0)\}\phi(x).1 - 0.$$

$$\phi'(x) = -1 - 4\cos x - 2\int_0^x \phi(\xi)d\xi + 3\phi(x) \quad —— (2)$$

Again, differentiating with respect to x, we have

$$\phi''(x) = 0 + 4\sin x - 2 \cdot \frac{\partial}{\partial x}\left\{\int_0^x \phi(\xi)\,d\xi\right\} + 3\phi'(x)$$

$$\phi''(x) = 4\sin x - 2\left[\int_0^x\left[\frac{\partial}{\partial x}\phi(\xi)\right]d\xi + \{\phi(\xi)\}_{\xi=x}\frac{d}{dx}\cdot x - \{\phi(\xi)\}_{\xi=0}\frac{d}{dx}\cdot 0\right] + 3\phi'(x)$$

$$\phi''(x) - 3\phi'(x) = 4\sin x - 2\left[0 + \phi(x) - 0\right]$$

$$\phi''(x) - 3\phi'(x) + 2\phi(x) = 4\sin x \qquad \text{—— (3)}$$

which is the original differential equation

Also we have

from (1) $\phi(0) = 1$

from (2) $\phi'(0) = -1 - 4 + 3\phi(0)$

$= -5 + 3$

$\phi'(0) = -2$

So, conditions are $\phi(0) = 1$ and $\phi'(0) = -2$.

which is the original differential equation with initial conditions.

(2) Reduce the initial value problem

$$\phi''(x) - \sin x \cdot \phi'(x) + e^{x}\phi(x) = x$$

with the initial conditions $\phi(0) = 1$, $\phi'(0) = -1$.

to a non homogeneous Volterra's integral equation of second kind. Conversely obtain the original differential equation with initial conditions from the obtained integral equation.

Solution. Given

$$\phi''(x) = x + \sin x\,\phi'(x) - e^{x}\phi(x) \qquad \text{—— (1)}$$

Initial conditions are $\phi(0) = 1$, $\phi'(0) = -1$ —— (2)

Integrating both sides of (1) with respect to x from $x=0$ to $x=x$, we have

$$\int_0^x \phi''(\xi)d\xi = \int_0^x \xi d\xi - \int_0^x e^{\xi}\phi(\xi)d\xi + \int_0^x \sin\xi\, \phi'(\xi)d\xi$$

$$\Rightarrow \phi'(x)-\phi'(0) = \frac{x^2}{2} - \int_0^x e^{\xi}\phi(\xi)d\xi + \left[\sin\xi \cdot \phi(\xi)\Big|_0^x - \int_0^x \cos\xi\cdot\phi(\xi)d\xi\right]$$

$$\Rightarrow \phi'(x)+1 = \frac{x^2}{2} - \int_0^x e^{\xi}\phi(\xi)d\xi + \sin x\cdot\phi(x) - 0 - \int_0^x \cos\xi\,\phi(\xi)d\xi$$

$$\Rightarrow \phi'(x) = -1+\frac{x^2}{2} + \sin x\cdot\phi(x) - \int_0^x (e^{\xi}+\cos\xi)\phi(\xi)\,d\xi \quad \text{— (3)}$$

Again, integrating both sides of (3) with respect to x from $x=0$ to $x=x$, we have

$$\int_0^x \phi'(\xi)\,d\xi = -\int_0^x d\xi + \frac{1}{2}\int_0^x \xi^2 d\xi + \int_0^x \sin\xi\cdot\phi(\xi)d\xi - \int_0^x (e^{\xi}+\cos\xi)\phi(\xi)d\xi^2$$

$$\phi(x)-\phi(0) = -x+\frac{1}{6}x^3 + \int_0^x \sin\xi\cdot\phi(\xi)d\xi - \int_0^x (x-\xi)(e^{\xi}+\cos\xi)\phi(\xi)d\xi$$

$$\phi(x) = 1-x+\frac{1}{6}x^3 + \int_0^x \left[\sin\xi-(x-\xi)(e^{\xi}+\cos\xi)\right]\phi(\xi)d\xi \quad -(4)$$

$[\because \phi(0)=1]$

which is non homogeneous Volterra's integral equation of second kind.

Converse. Suppose

$$\phi(x) = 1-x+\frac{1}{6}x^3 + \int_0^x \left[\sin\xi-(x-\xi)(e^{\xi}+\cos\xi)\right]\phi(\xi)d\xi \quad \text{— (5)}$$

$$\Rightarrow \phi(0) = 1-0+0+0 \Rightarrow \phi(0)=1 \quad \text{— (6)}$$

Now, Differentiating (5) with respect to x, we have

$$\phi'(x) = 0-1+\frac{3x^2}{6} + \frac{\partial}{\partial x}\left[\int_0^x \left[\sin\xi-(x-\xi)(e^{\xi}+\cos\xi)\right]\phi(\xi)d\xi\right]$$

$$\Rightarrow \phi'(x) = -1+\frac{x^2}{2} + \int_0^x \frac{\partial}{\partial x}\left[\left[\sin\xi-(x-\xi)(e^{\xi}+\cos\xi)\right]\cdot\phi(\xi)\right]d\xi$$

$$+\left[\left[\sin\xi-(x-\xi)(e^{\xi}+\cos\xi)\right]\phi(\xi)\right]_{\xi=x}\frac{d}{dx}x - 0$$

$$\phi'(x) = -1 + \frac{x^2}{2} + \int_0^x \left[0 - (1-0)(e^{\xi} + \cos\xi)\right] \phi(\xi)\, d\xi$$
$$+ \left[\sin x - (x-x)(e^x + \cos x)\right] \phi(x)$$

$$\phi'(x) = -1 + \frac{x^2}{2} - \int_0^x (e^{\xi} + \cos\xi)\phi(\xi)\, d\xi + \sin x \cdot \phi(x) \quad -(7)$$

Observe Here $\phi'(0) = -1$ (8)

Again, differentiating, we have

$$\phi''(x) = 0 + \frac{2x}{2} - \frac{\partial}{\partial x}\left[\int_0^x (e^{\xi} + \cos\xi)\phi(\xi)\, d\xi\right] + \sin x\, \phi'(x)$$
$$+ \cos x \cdot \phi(x)$$

$$\phi''(x) = x - \int_0^x \frac{\partial}{\partial x}\left\{(e^{\xi} + \cos\xi)\phi(\xi)\right\} \cdot d\xi$$
$$- \left[(e^{\xi} + \cos\xi)\phi(\xi)\right]_{\xi = x} \frac{d}{dx} \cdot x + 0 + \sin x\, \phi'(x) + \cos x \cdot \phi(x)$$

$$\phi''(x) = x - 0 - (e^x + \cos x)\phi(x) + \sin x \cdot \phi'(x) + \cos x \cdot \phi(x)$$
$$= x - e^x \phi(x) - \cancel{\cos x\, \phi(x)} + \sin x \cdot \phi'(x) + \cancel{\cos x \cdot \phi(x)}$$

$$\phi''(x) = x - e^x \phi(x) + \sin x\, \phi'(x)$$

$$\Rightarrow \phi''(x) - \sin x\, \phi'(x) + e^x \phi(x) = x$$

with initial conditions $\phi(0) = 1$, $\phi'(0) = -1$

which is the original differential equation with initial conditions.

Proved.

(3) Convert the differential equation $\frac{d^2\phi}{dx^2} - 2x\frac{d\phi}{dx} - 3\phi = 0$ with initial conditions $\phi(0)=0$, $\phi'(0)=0$ to Volterra integral equation of second kind. Conversely obtain the original differential equation with the initial conditions from the obtained integral equation.

Solution. Given

$$\frac{d^2\phi}{dx^2} - 2x\frac{d\phi}{dx} - 3\phi = 0 \quad \text{——} (1)$$

with initial conditions
$$\left.\begin{aligned}\phi(0)&=0\\ \phi'(0)&=0\end{aligned}\right\} \quad \text{——} (2)$$

Integrating (1) from $x=0$ to $x=x$, we have

$$\int_0^x \frac{d^2\phi}{d\xi^2}d\xi - 2\int_0^x \xi\frac{d\phi}{d\xi}d\xi - 3\int_0^x \phi(\xi)d\xi = 0$$

$$\Rightarrow \left[\frac{d\phi}{d\xi}\right]_0^x = 2\int_0^x \xi\frac{d\phi}{d\xi}d\xi + 3\int_0^x \phi(\xi)d\xi$$

$$\Rightarrow \frac{d\phi}{dx} - \left(\frac{d\phi}{dx}\right)_{x=0} = 2\left[\xi\cdot\phi(\xi)\Big|_0^x - \int_0^x 1.\,\phi(\xi)d\xi\right] + 3\int_0^x \phi(\xi)d\xi$$

$$\Rightarrow \frac{d\phi}{dx} - \phi'(0) = 2[x\phi(x) - 0] - 2\int_0^x \phi(\xi)d\xi + 3\int_0^x \phi(\xi)d\xi$$

$$\Rightarrow \frac{d\phi}{dx} = 2x\phi(x) + \int_0^x \phi(\xi)d\xi \quad [\because \phi'(0)=0]$$

$$\Rightarrow \phi'(x) = 2x\phi(x) + \int_0^x \phi(\xi)d\xi \quad \text{——} (3)$$

Again, Integrating from $x=0$ to $x=x$, We have

$$\int_0^x \phi'(\xi)\,d\xi = 2\int_0^x \xi\,\phi(\xi)\,d\xi + \int_0^x \phi(\xi)\,d\xi^2$$

$$\Rightarrow \phi(x) - \phi(0) = 2\int_0^x \xi\,\phi(\xi)\,d\xi + \int_0^x (x-\xi)\,\phi(\xi)\,d\xi$$

$$= \int_0^x (2\xi + x - \xi)\,\phi(\xi)\,d\xi$$

$$\Rightarrow \phi(x) = \int_0^x (x+\xi)\,\phi(\xi)\,d\xi \quad \text{--- (4)}$$

$$[\because \phi(0) = 0]$$

which is volterra's integral equation of second kind.

Converse. We have

$$\phi(x) = \int_0^x (x+\xi)\,\phi(\xi)\,d\xi \quad \text{--- (5)}$$

Here $\phi(0) = 0$ — (6)

$$\left[\because \int_0^0 (x+\xi)\phi(\xi)\,d\xi = 0\right]$$

Differentiating (5), we have

$$\phi'(x) = \frac{\partial}{\partial x}\left[\int_0^x (x+\xi)\,\phi(\xi)\,d\xi\right]$$

$$= \int_0^x \left\{\frac{\partial}{\partial x}\left[(x+\xi)\phi(\xi)\right]\right\} \cdot d\xi$$

$$+ \left[(x+\xi)\phi(\xi)\right]_{\xi=x} \frac{d}{dx}x - \left[(x+\xi)\phi(\xi)\right]_{\xi=0} \frac{d}{dx}\cdot 0$$

$$\phi'(x) = \int_0^x \phi(\xi)\,d\xi + 2x\,\phi(x) - 0$$

$$\phi'(x) = \int_0^x \phi(\xi)\,d\xi + 2x\,\phi(x) \quad \text{--- (6)}$$

Here $\phi'(0) = 0$ — (7)

Again, differentiating, we have

$$\phi''(x) = \frac{\partial}{\partial x}\left[\int_0^x \phi(\xi)\,d\xi\right] + 2\left[1\cdot\phi(x) + x\cdot\phi'(x)\right].$$

$$= \int_0^x \frac{\partial}{\partial x}(\phi(\xi))\cdot d\xi + \phi(\xi)\Big|_{\xi=x}\cdot\frac{d}{dx}x - [\phi(\xi)]_{\xi=0}\frac{d}{dx}\cdot 0$$

$$+\, 2\phi(x) + 2x\phi'(x)$$

$$\phi''(x) = 0 + \phi(x) - 0 + 2\phi(x) + 2x\phi'(x)$$

$$\Rightarrow \phi''(x) - 3\phi(x) - 2x\phi'(x) = 0$$

$$\Rightarrow \phi''(x) - 2x\phi'(x) - 3\phi(x) = 0$$

with initial conditions $\phi(0)=0$, $\phi'(0)=0$ } initial, original differential equation.

(4) Reduce

$$\phi''(x) + \lambda\phi(x) = F(x) \text{ where } \phi(0)=1,\ \phi'(0)=0$$

Soln. Given $\phi''(x) = F(x) - \lambda\phi(x)$ —— (1)

Integrating, we have

$$\phi'(x) - \phi'(0) = \int_0^x F(\xi)\,d\xi - \lambda\int_0^x \phi(\xi)\,d\xi$$

$$\Rightarrow \phi'(x) = \int_0^x F(\xi)\,d\xi - \lambda\int_0^x \phi(\xi)\,d\xi \quad [\because \phi'(0)=0$$

Again, integrating, we have

$$\int_0^x \phi'(\xi)\,d\xi = \int_0^x [F(\xi) - \phi(\xi)]\,d\xi$$

$$\phi(x) - \phi(0) = \int_0^x [F(\xi) - \phi(\xi)]\,d\xi^2$$

$$\Rightarrow \phi(x) = 1 + \int_0^x (x-\xi)(F(\xi) - \phi(\xi))\,d\xi$$

which is reduced Volterra integral equation.