For M.Sc third semester

COMPLEX NUMBERS

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Complex Analysis

The solutions of the eqs. $x^2-1=0$ do not exist in the set of real numbers. So it was required to extend the system of numbers. Euler was first to introduce the symbol i such that $i^2=-1$. Later Gauss and Hamilton gave significant contributions in the development of the theory of complex numbers.

Complex Number: If we represent a complex number by z then it is defined as an ordered pair $z=(x,y)$, where x, y are real numbers. We can also write $z = x+iy$, where $i=\sqrt{-1}$
x — real part of z and y = imaginary part of z.

* Every real number is also a complex number whose imaginary part is zero.

* Two complex numbers $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$ are equal iff their real parts and imaginary parts are equal i.e iff $x_1=x_2$ and $y_1=y_2$.

* Conjugate of a complex number $z=x+iy$ is denoted by $\bar{z}$ and $\bar{z} = x-iy$. [Replace i by $-i$]

* Modulus of a complex number $z=x+iy$ is denoted by $|z|$ and $|z| = \sqrt{x^2+y^2}$

We have studied various algebraic properties of complex numbers in detail in our U.G. Course. Here we have to refresh them only.

Complex functions: If we are able to find one or more complex number w for every value of z in a definite domain D, then w is said to a function of z defined over domain D. We represent $w = f(z)$ where $z = x+iy$. Since w is also complex we write $w = u+iv$ where u and v functions of two real variables x and y.

We can write $w = u(x,y) + i\,v(x,y) = f(x+iy)$
where x and y are real numbers.

* We can represent w and z on Argand plane.

Neighbourhood of a point: A neighbourhood of a complex number z_0 in the Argand plane is defined as an open disc $|z - z_0| < \delta$, where δ is an arbitrary small +ive real number. It is called the radius of this neighbourhood.

Limit: In a closed and bounded domain D, the number l is said to be the limit of $f(z)$ as z approaches a along any path in D if for a given $\epsilon > 0$, $\exists\ \delta > 0$ s.t. $|f(z) - l| < \epsilon$ whenever $|z - a| < \delta$. We represent it as

$$\lim_{z \to a} f(z) = l$$

Continuity: In a closed and bounded domain D, a complex function $f(z)$ of z is said to be continuous at $a \in D$ if for a given $\epsilon > 0$, $\exists\ \delta > 0$ s.t. $|f(z) - f(a)| < \epsilon$ whenever $|z - a| < \delta$.

It is obvious that if $\lim_{z \to a} f(z) = f(a)$ then the function $f(z)$ is continuous at $z = a$.

Uniform Continuity: A function $f(z)$ defined in a closed and bounded domain D is said to be uniformly continuous in D if for a given $\epsilon > 0$ $\exists$ a $\delta > 0$ s.t. $|f(z_1) - f(z_2)| < \epsilon$ whenever $|z_1 - z_2| < \delta$. where $z_1, z_2 \in D$.

* Uniform continuity is the property associated with the domain not with a point.

* If a function $f(z)$ is uniformly continuous in a domain D then it is continuous at each point of D.

* Differentiability: Let $f(z) = w$, be a function of z defined in D, then $f(z)$ is said to be differentiable at $z_0 \in D$ iff

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$ exists uniquely and finite. It is denoted by $f'(z_0)$. If this limit is not unique then the derivative at z_0 does not exist.

* Continuity is a necessary but not a sufficient condition for differentiability.
i.e. if a function is differentiable then it is always continuous; but converse may not be true. e.g. $f(z) = |z|^2$ is continuous everywhere but is not differentiable anywhere.