For M.Sc third semester

ANALYTIC FUNCTIONS

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Analytic function: A single valued function $f(z)$ defined over a domain D is said to be analytic at a point $z_0 \in D$ if it is differentiable not only at z_0 but also in some neighbourhood of z_0. An analytic function is also called as holomorphic or regular function.

Necessary Condition for a function $f(z)$ to be analytic

~~Let~~ = If a function $f(z) = u(x,y) + i\, v(x,y)$ is differentiable at any point $z = x+iy$ then the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ should exist and satisfy the equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. These two equations are called as Cauchy- Riemann equations.

Proof: Let $w = f(z) = u(x,y) + i\,v(x,y)$ where — (1)
$z = x+iy$ then $\Delta z = \Delta x + i\,\Delta y$.

Since the function is differentiable at any z, so

$$\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \text{ must exist}$$

uniquely as $\Delta z \to 0$ through any path

Using Eq. (1)

$$\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to 0} \left[\frac{u(x+\Delta x, y+\Delta y) - u(x,y)}{\Delta x + i\,\Delta y} + i \left[\frac{v(x+\Delta x, y+\Delta y) - v(x,y)}{\Delta x + i\Delta y}\right] \right] \quad \text{— (2)}$$

we choose the path $\Delta y = 0$

Equation (2) gives

$$\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta x \to 0} \left[\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \left[\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \right] \right]$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \qquad \text{— (3)}$$

[As $f(z)$ is differentiable the partial derivatives must exist]

Again choosing the path $\Delta x \to 0$, Equation (2) gives

$$\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta y \to 0} \left[\frac{u(x, y+\Delta y) - u(x,y)}{i \Delta y} + i \left[\frac{v(x, y+\Delta y) - v(x,y)}{i \Delta y} \right] \right]$$

$$= \left[\frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \right]$$

$$= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad \text{— (4)}$$

As derivative of a function is unique, Equations (3) & (4) give

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Comparing real and imaginary parts.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Which can also be written as

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

Which are required Cauchy – Riemann equations

Sufficient Condition for a function $f(z)$ to be analytic

The continuous single valued function $f(z) = u(x,y) + i\,v(x,y)$ is analytic in a domain D if the four partial derivatives $u_x = \frac{\partial u}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$, $v_x = \frac{\partial v}{\partial x}$, $v_y = \frac{\partial v}{\partial y}$ exist and satisfy Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ at each point of domain.

Proof: Before the proof of the result, we should be familiarized with the following results.

* A function which is continuous in a closed and bounded domain is uniformly continuous in domain
* Lagrange's Mean value theorem, which states that if $f(x)$ is defined and continuous in $[a, a+h]$ and differentiable in $]a, a+h[$ then $\exists$ a point $c = a+\theta h$, $0<\theta<1$ in $]a, a+h[$ such that

$$\boxed{f(a+h) - f(a) = h\, f'(a+\theta h)}$$

Let $w = f(z) = u(x,y) + i\,v(x,y)$

We know $u = u(x,y)$ so $u + \Delta u = u(x+\Delta x, y+\Delta y)$

So $\Delta u = u(x+\Delta x, y+\Delta y) - u(x,y)$

$$= u(x+\Delta x, y+\Delta y) - u(x+\Delta x, y) + u(x+\Delta x, y) - u(x,y)$$

$$= \Delta y\, u_y(x+\Delta x, y+\theta_1 \Delta y) + \Delta x\, u_x[x+\theta_2 \Delta x, y]$$

[By Mean value theorem]

where $0<\theta_1<1$ and $0<\theta_2<1$ —— ①

Since u_x and u_y are continuous functions in D, they are uniformly continuous in D. We have

$$\left.\begin{aligned} |u_y(x+\Delta x, y+\theta_1 \Delta y) - u_y(x,y)| &< \epsilon_1 \\ |u_x(x+\theta_2 \Delta x, y) - u_x(x,y)| &< \epsilon_1 \end{aligned}\right\} \text{--- (2)}$$

whenever $|\Delta x| < \delta$ and $|\Delta y| < \delta$

Let $u_y(x+\Delta x, y+\theta_1 \Delta y) - u_y(x,y) = \alpha_1$

and $u_x(x+\theta_2 \Delta x, y) - u_x(x,y) = \beta_1$

then from (2) $|\alpha_1| < \epsilon_1$ and $|\beta_1| < \epsilon_1$

from Eq. (1) we obtain

$$\Delta u = \{\alpha_1 + u_y(x,y)\}\Delta y + \{\beta_1 + u_x(x,y)\}\Delta x$$

Proceeding in the same manner, we get

$$\Delta v = \{\alpha_2 + v_y(x,y)\}\Delta y + \{\beta_2 + v_x(x,y)\}\Delta x$$

where $|\alpha_2| < \epsilon_2$ and $|\beta_2| < \epsilon_2$

Now $$\frac{\Delta f}{\Delta z} = \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}$$

$$= \frac{(u_y \Delta y + u_x \Delta x + \alpha_1 \Delta y + \beta_1 \Delta x) + i[v_y \Delta y + v_x \Delta x + \alpha_2 \Delta y + \beta_2 \Delta x]}{\Delta x + i\Delta y}$$

[$u_y = u_y(x,y)$ and so on]

$$= \frac{-v_x \Delta y + u_x \Delta x + i u_x \Delta y + i v_x \Delta x + \alpha_1 \Delta y + \beta_1 \Delta x + i\alpha_2 \Delta y + i\beta_2 \Delta x}{\Delta x + i\Delta y}$$

So $\dfrac{\Delta f}{\Delta z} = \dfrac{(u_x + i v_x)(\Delta x + i\Delta y) + \alpha_1 \Delta y + \beta_1 \Delta x + i\alpha_2 \Delta y + i\beta_2 \Delta x}{\Delta x + i\Delta y}$

$$= u_x + i v_x + \frac{(\alpha_1 + i\alpha_2)\Delta y}{\Delta x + i\Delta y} + \frac{(\beta_1 + i\beta_2)\Delta x}{\Delta x + i\Delta y}$$

Or $\left|\dfrac{\Delta f}{\Delta z} - u_x + i v_x\right| = \left|\dfrac{(\alpha_1 + i\alpha_2)\Delta y}{\Delta x + i\Delta y} + \dfrac{(\beta_1 + i\beta_2)\Delta x}{\Delta x + i\Delta y}\right|$

$$\leq \left|\frac{(\alpha_1 + i\alpha_2)\Delta y}{\Delta x + i\Delta y}\right| + \left|\frac{(\beta_1 + i\beta_2)\Delta x}{\Delta x + i\Delta y}\right|$$

$$\leq \frac{|\alpha_1 + i\alpha_2|\,|\Delta y|}{|\Delta x + i\Delta y|} + \frac{|\beta_1 + i\beta_2|\,|\Delta x|}{|\Delta x + i\Delta y|}$$

$$\leq |\alpha_1| + |\alpha_2| + |\beta_1| + |\beta_2| \quad \left[\begin{array}{l}\text{for } |\Delta y| \leq |\Delta x + i\Delta y| \\ \text{and } |\Delta x| \leq |\Delta x + i\Delta y|\end{array}\right]$$

$$\leq 2(\epsilon_1 + \epsilon_2)$$

Hence $\lim\limits_{\Delta z \to 0} \dfrac{\Delta f}{\Delta z} = u_x + i v_x$

Or $f'(z) = u_x + i v_x$

Since u_x and v_x exist, the derivative of the function $f(z)$ exists and hence the function $f(z)$ is an analytic function.

Polar form of Cauchy - Riemann Equations

Cauchy - Riemann Eqs. in cartesian form are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad — (1)$$

* We know the relations between cartesian and polar coordinates are

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

$$\therefore \quad x^2+y^2 = r^2 \; — (1) \quad \text{and} \quad \theta = \tan^{-1}(y/x) \; — (2)$$

Differentiating (1) & (2) partially w.r.t x and y

$$2r\frac{\partial r}{\partial x} = 2x = 2r\cos\theta$$

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta} \quad \text{and} \quad \boxed{\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta}$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2} = -\frac{r\sin\theta}{r^2}$$

$$\boxed{\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}} \qquad \frac{\partial \theta}{\partial y} = \frac{1}{1+(y/x)^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2}$$

$$\boxed{\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}}$$

Now
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial r}\cos\theta + \frac{\partial u}{\partial \theta}\left(-\frac{\sin\theta}{r}\right)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} = \frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\left(\frac{\cos\theta}{r}\right)$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} = \frac{\partial v}{\partial r}\cdot\cos\theta + \frac{\partial v}{\partial \theta}\left(-\frac{\sin\theta}{r}\right)$$

and
$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} = \frac{\partial v}{\partial r}\cdot\sin\theta + \frac{\partial v}{\partial \theta}\left(\frac{\cos\theta}{r}\right) \quad — (2)$$

Substitute the values from equations (2) in equations (1)

$$\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r} = \frac{\partial v}{\partial r}\sin\theta + \frac{\partial v}{\partial \theta}\frac{\cos\theta}{r} \quad \text{--- (3)}$$

and
$$\frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r} = -\frac{\partial v}{\partial r}\cos\theta + \frac{\partial v}{\partial \theta}\frac{\sin\theta}{r} \quad \text{--- (4)}$$

Now (3) $\times \cos\theta$ + (4) $\times \sin\theta$, we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \longrightarrow (5)$$

And (3) $\times \sin\theta$ − (4) $\times \cos\theta$, we get

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r} \longrightarrow (6).$$

Equations (5) and (6) are the required form of Cauchy – Riemann equations.