UNIT-4

Hamiltonian Canonical Eq. or Hamiltonian Equation of Motion:-

Since the Hamiltonian is the function of (p_j, q_j, t) then it can be expressed as

$$H = H(p_j, q_j, t)$$

on differentiating we get

$$\partial H = \sum_j \frac{\partial H}{\partial p_j} dp_j + \sum_j \frac{\partial H}{\partial q_j} dq_j + \sum_j \frac{\partial H}{\partial t} dt \quad \text{--- (1)}$$

The Relation b/w Hamiltonian And lagrangian can be Expressed as

$$H = \sum_j p_j \dot{q}_j - L$$

differentiating we get.

$$\partial H = \sum_j \dot{q}_j dp_j + \sum_j p_j d\dot{q}_j - dL \quad \text{--- (2)}$$

Since lagrangian is the function of q_j And it's Time derivative $\dot{q}_j$ Then it can be Expressed as

$$L = L(q_j, \dot{q}_j, t)$$

differentiating we get

$$\partial L = \sum_j \frac{\partial L}{\partial q_j} dq_j + \sum_j \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j + \sum_j \frac{\partial L}{\partial t} dt \quad \text{--- (3)}$$

Substituting the Value of dL in Eq (2) we get

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$$dH = \sum_j \dot{q}_j dp_j + \sum_j p_j d\dot{q}_j - \sum_j \frac{\partial L}{\partial q_j} dq_j - \sum_j \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j$$

$$- \sum_j \frac{\partial L}{\partial t} dt$$

Now Consider the Term which represent the Relation b/w lagrangian momentum Coordinate And position Coordinate Can be written as

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

$$\text{And } \dot{p}_j = \frac{\partial L}{\partial q_j}$$

Now above equation becomes

$$dH = \sum_j \dot{q}_j dp_j + \cancel{\sum_j p_j d\dot{q}_j} - \sum_j \dot{p}_j dq_j - \cancel{\sum_j p_j d\dot{q}_j} - \sum_j \frac{\partial L}{\partial t} dt$$

$$dH = \sum_j \dot{q}_j dp_j - \sum_j \dot{p}_j dq_j - \sum_j \frac{\partial L}{\partial t} dt \quad \text{--- (4)}$$

On Comparing Eq (1) & Eq (4) we get

$$\left.\begin{aligned} \boxed{\dot{p}_j = -\frac{\partial H}{\partial q_j}} \\ \boxed{\dot{q}_j = \frac{\partial H}{\partial p_j}} \end{aligned}\right\} \quad \text{--- (5)}$$

$$\boxed{\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}} \quad \text{--- (6)}$$

Since Eq (6) is known as Hamiltonian Canonical Eq.

* Prove that in the cyclic coordinate the momentum remains conserved

A coordinate is said to be cyclic if it does not appear in the Lagrangian of the system.

It can be expressed as

$$\frac{\partial L}{\partial q_k} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0$$

Now the Lagrangian Eq of the cyclic coordinate can be expressed as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0 \quad (1)$$ NB

On integrating we get

$$\frac{\partial L}{\partial \dot{q}_k} = \text{Constant} \quad (2)$$ NB

$$\frac{\partial (T-V)}{\partial \dot{q}_k} = \text{Constant}$$

We know that the Lagrangian of any system is the difference of K.E and P.E i.e

$$L = T - V$$

But for a conservative system

$$V = 0$$

$$\frac{\partial T}{\partial \dot{q}_k} = \text{Constant}$$

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If the particle is moving along X- Axis then K.E can be written as

$$T = \frac{1}{2} m \dot{x}^2$$

diff w.r.to $\dot{x}$

$$\frac{\partial T}{\partial \dot{x}} = \frac{1}{2} m \times 2\dot{x}$$

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x} = p = \text{constant}$$

It means that for cyclic coordinates the momentum remains always conserved.

Note point:-

$$H = \sum_j p_j \dot{q}_j - L$$

$\cancel{p = mv}$

$$v = \frac{dq}{dt} = \dot{q}$$

$$v = \dot{q}$$

$$v_j = \dot{q}_j$$

$$T = \frac{1}{2} m v_j^2$$

$$[m v_j^2 = 2T]$$

$$H = \sum_j m v_j \times v_j - L$$

$$H = \sum_j m v_j^2 - L$$

$$H = 2T - L$$

$$H = 2T - [T - V]$$

$$H = 2T - T + V$$

$$\boxed{H = T + V}$$

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* If a Lagrangian can be Expressed in the form of $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$

Then Calculate the Cyclic Coordinate And show the Corresponding ~~Eq~~ Conservation law.

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

In the above Condition θ is Not Present then θ is known as Cyclic Coordinate. The mathematical form can be Expressed as

$$\frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial q_k} = 0$$

$$q_k = \theta \quad \theta \quad — (1)$$

Now the Condition of Lagrangian in Term of Coordinate θ

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

Since $\frac{\partial L}{\partial \theta} = 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0 \quad — (2)$$

Now we have,

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

~~Partialy diff w.r.to $\dot{\theta}$ we get~~

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0$$

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$$\frac{d}{dt}(I\omega) = 0$$

$$\dot{J} = 0$$

$$\boxed{J = \text{Constant}}$$

From above, it is clea that when the Co-ordinate θ is Not present i.e. θ is Cyclic Co-ordinate then Angular Momentum always remains Constant