* Motion under Inverse square force: And Conic path: Kepler's problem
Energy And Eccentricity -

Those force whose magnitude depend upon inverse of square of distance from the fixed point is known as inverse square force And law is know as inverse square law.
According to the definition of Inverse square force we know that

$$F(r) \propto \frac{1}{r^2}$$

$$F(r) = \frac{-K}{r^2} \quad \text{—(1)}$$

$$F(r) = \frac{dV}{dr}$$

$$-\frac{dV}{dr} = \frac{-K}{r^2}$$

$$\boxed{V(r) = \frac{-K}{r}}$$

Equation of orbits in the form of polar

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2u^2} f\left(\frac{1}{u}\right) \qquad f\left(\frac{1}{u}\right) = -Ku^2$$

$$\frac{d^2u}{d\theta^2} + u = \frac{m \times K u^2}{l^2u^2}$$

$$\frac{d^2u}{d\theta^2} + u - \frac{mK}{l^2} = 0 \quad \text{—(2)}$$

put $u - \frac{mK}{l^2} = y$ constants

$$\frac{d^2u}{d\theta^2} + y = 0 \quad \text{—(3)}$$

Scanned with CamScanner

This an differential Equation So The General soln of This differential Eq Can be written as in the form of

$$y = u' \cos\theta \quad — (4)$$

$$u - \frac{mk}{l^2} = u' \cos\theta$$

$$u = u' \cos\theta + \frac{mk}{l^2}$$

$$\frac{1}{r} = \frac{u' l^2 \cos\theta + mk}{l^2}$$

$$\boxed{\frac{1}{r} = \frac{\frac{u' l^2 \cos\theta}{mk} + 1}{l^2 / mk}}$$

Lets $\frac{u' l^2}{mk} = \varepsilon$ (Eccentricity

$$l^2 / mk = p$$

$$\frac{1}{r} = \frac{1 + \varepsilon \cos\theta}{p}$$

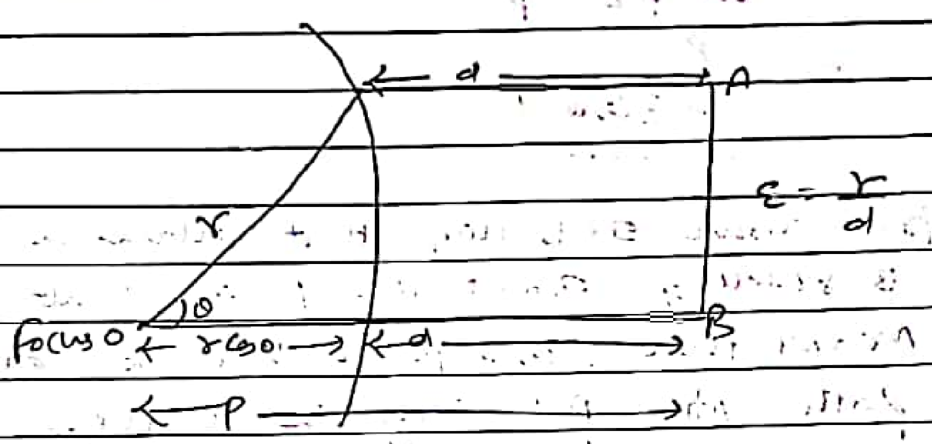
$$\boxed{r = \frac{p}{1 + \varepsilon \cos\theta}}$$

Hence we can say that Any particle which is moving under inverse square force show Conic path. And path of particle depend upon the Value of e

If $e = 0$ Than the path of particle is circle

If $e > 1$ then the path of the particle is hyperbola

$e < 1 =$ the path will be Ellipse



Now Let us Consider a particle which is rotating about a fix point O. Let r is Radius of The particle And it makes Angle θ with the fixed Axis

Now the Eccentricity of Any particle from the fixed point And the distance of the particle from the point of measurement

$$\varepsilon = \frac{r}{d}$$

$$d = \frac{r}{\varepsilon}$$

$$P = r\cos\theta + d$$

$$P = r\cos\theta + \frac{r}{\varepsilon}$$

$$P = \frac{r\varepsilon\cos\theta + r}{\varepsilon}$$

$$P = \frac{r(\varepsilon\cos\theta + 1)}{\varepsilon}$$

$$\frac{P}{1+\varepsilon\cos\theta} = \frac{r}{\varepsilon}$$

$$r = \frac{P\varepsilon}{1+\varepsilon\cos\theta}$$

Scanned with CamScanner

Let $p\varepsilon = p$

$$\boxed{r = \frac{p}{1+\varepsilon\cos\theta}}$$

from above it is clear that when a particle is rotating about the fixed point it means that it's path will be conic path And path may be circular, elliptical, hyperbolic And parrabolic.

Relation b/w Energy And Eccentricity :-

We know that the total Energy of the particle can be written as

$$E = \frac{1}{2} m\dot{r}^2 + \frac{l^2}{2mr^2} + V(r) \quad \text{--- (5)}$$

According to the inverse square law potential energy of the particle can be written as

$$E = \frac{1}{2} m\dot{r}^2 + \frac{l^2}{2mr^2} - \frac{k}{r} \quad \text{--- (6)}$$

We know that

$$r = \frac{p}{1+\varepsilon\cos\theta}$$

$$r = \frac{p}{1+\varepsilon} \quad \left\{ \text{for minimum value } \theta = 0 \right\}$$

If the radius is fix then $\dot{r}$ will be zero

$$E = \frac{l^2(1+\varepsilon)^2}{2mp^2} - \frac{k(1+\varepsilon)}{p}$$

we know that

$$p = \frac{l^2}{mK}$$ Then above Equation

becomes

~~$E = \ldots$~~

$$E = \frac{l^4 (1+\varepsilon)^2 m^4 K^2}{2m\, l^4\, l^2} - \frac{K(1+\varepsilon)\cdot mK}{l^2}$$

$$E = \frac{mK^2 (1+\varepsilon)^2}{2l^2} - \frac{mK^2 \cdot (1+\varepsilon)}{l^2}$$

$$E = \frac{mK^2 (1+\varepsilon)}{l^2} \left\{ \frac{1+\varepsilon}{2} - 1 \right\}$$

$$E = \frac{mK^2}{l^2} (1+\varepsilon) \left(\frac{\varepsilon - 1}{2} \right)$$

$$E = \frac{mK^2}{2l^2} (\varepsilon^2 - 1)$$

$$\frac{2l^2 E}{mK^2} = \varepsilon^2 - 1$$

$$\varepsilon^2 = 1 + \frac{2l^2 E}{mK^2}$$

$$\varepsilon = \sqrt{1 + \frac{2l^2 E}{mK^2}}$$ putting this in Conic path

$$\boxed{r = \frac{\frac{l^2}{mK}}{1 + \left(1 + \frac{2l^2 E}{mK^2}\right)^{1/2} \cos\theta}}$$

from above eq it is clear that this is

Known as Kepler first law And also Known as Kepler law of planetary motion.

* Kepler Secound law:-
Kepler secound law of Arial Velocity:-

Let us describe the position of the particle in the plane polar coordinate ~~Co~~ r And θ Lagrangian will be

$$L = T - V$$

$$T = \frac{1}{2} m[\dot{r}^2 + r^2 \dot{\theta}^2]$$

Now from Lagrange

$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \quad \text{--- (1)}$$

~~Lagrangian Eq of motion in~~ Terms of ~~r And θ can be written as~~

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad \text{--- (2)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \quad \text{--- (3)}$$

~~Now from Eq (1)~~

$$\cancel{L = \frac{1}{2} m[\dot{r}^2 + r^2 \dot{\theta}^2] - V(r)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

diff above Equation w r to $\frac{\partial L}{\partial \dot{\theta}}, \frac{\partial L}{\partial \theta}, \frac{\partial L}{\partial \dot{r}}, \frac{\partial L}{\partial r}$ we get

$$\frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{dV}{dr}$$

Now $\frac{d}{dt}(mr^2\dot{\theta}) = 0$ (from Eq (2))

~~now from Eq (3) we have~~

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$\frac{d}{dt}(m\dot{r}) - \left[mr\dot{\theta}^2 - \frac{dV}{dr}\right] = 0$$

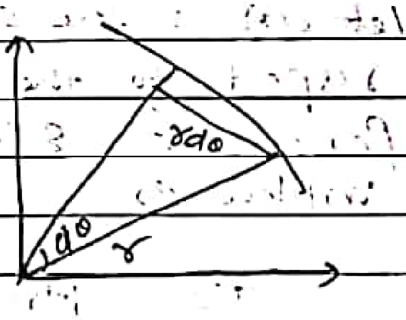
$$(m\ddot{r}) - \left[mr\dot{\theta}^2 - \frac{dV}{dr}\right] = 0 \quad (4)$$

$$mr^2\dot{\theta} = \text{constant}$$

$$\frac{1}{2} r^2\dot{\theta} = \text{constant}$$

$$\left\{\text{Because } m = \frac{1}{2}\right\}$$

Since m is constant the factor $\frac{1}{2}$ inserting in above Eq so that $\frac{1}{2} r^2 \dot{\theta}$ represents the Arial velocity.



Let us consider a radius r and Angle θ, dA is the Area swept out by the Radius Vector in a Time dt Then

$$dA = \frac{1}{2} r^2 \, d\theta$$

And the Rate at which this Area is swept out will be

Scanned with CamScanner

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta}$$

$$\boxed{\frac{dA}{dt} = \text{Constant}}$$

from above it is clear that the velocity of The particle moving in a constant Area is always constant This is also known as Kepler second law of motion.

* Kepler's Third law
or Period of Eliptical orbit :-

Period of Eliptical orbit is the ratio of Area of Ellipse to the Rate of Area swept out
Let dA is the swept out Area with respect to the Time dt Then the Period of Eliptical orbit (T) may be written as

$$T = \frac{\text{Area of Ellipse}}{dA/dt} \quad \text{—(1)}$$

We know that swept out Area
i.e. $\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta}$ —(2)

Angular momentum

$$l = mr^2 \dot{\theta}$$

$$\dot{\theta} = \frac{l}{mr^2}$$

Scanned with CamScanner

put in Eq (2)

$$\frac{dA}{dt} = \frac{l}{2m}$$

$$T = \frac{\pi ab \times 2m}{l} \quad \text{—— (3)}$$

where ab are the semi major Axis whose value is given that:-

$$b = a\sqrt{1-\varepsilon^2}$$

$$a = \frac{P}{1-\varepsilon^2} = \frac{l^2/mK}{1-\varepsilon^2}$$

$$a = \frac{l^2/mK}{1-\varepsilon^2}$$

$$1-\varepsilon^2 = \frac{l^2}{mKa}$$

$$T = \frac{\pi a^2 \sqrt{1-\varepsilon^2} \times 2m}{l}$$

$$T^2 = \frac{\pi^2 a^4 (1-\varepsilon^2) \times 4m^2}{l^2}$$

$$T^2 = \frac{\pi^2 a^4 4m^2}{l^2} \times \frac{l^2}{mKa}$$

$$T^2 = \frac{4\pi^2 m a^3}{K}$$

$$\boxed{T^2 \propto a^3}$$

from above eq it's clear that square of Period of Revolution is proportional to the Cube of semi major Axis