* Lagrange's Equation from Hamilton Principle :-

We know that the Lagrangian L is a function of Generalized Velocities $\dot{q}_k$'s and Generalized Co-ordinates and time t i.e

$$L = L(q_1, q_2, q_3 - q_k, \cdots q_n, \dot{q}_1, \dot{q}_2 \cdots \dot{q}_k \cdots \dot{q}_n, t)$$

If the Lagrangian does not depend upon the time then the Variation δL can be written as

$$\delta L = \sum_{k=1}^{n} \frac{\partial L}{\partial q_k} \delta q_k + \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \quad \text{--- (1)}$$

Now Integrating both side from $t = t_1$ to $t = t_2$ we get

$$\int_{t_1}^{t_2} \delta L \, dt = \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k \, dt + \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \, dt$$

But According to the Hamilton Principle

$$\delta \int_{t_1}^{t_2} L \, dt = 0$$

$$0 = \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k \, dt + \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \, dt$$

where $\delta \dot{q}_k = \frac{d}{dt}(\delta q_k)$

$$\underbrace{\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k \, dt}_{\mathrm{I}} + \underbrace{\int_{t_1}^{t_2} \sum_k \underbrace{\frac{\partial L}{\partial \dot{q}_k}}_{\mathrm{I}} \underbrace{\delta \dot{q}_k}_{\mathrm{II}} \, dt}_{\mathrm{II}} = 0 \quad \text{--- (2)}$$

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Now Integrating the Secound Term of the Equation (2) by parts we get

$$\int_{t_1}^{t_2} \sum_K \frac{\partial L}{\partial \dot{q}_K} \delta \dot{q}_K \, dt = \sum \left[\frac{\partial L}{\partial \dot{q}_K} \delta q_K \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_K \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_K}\right) \delta q_K \, dt \quad \text{--- (3)}$$

Now at the End points of the path at the time t_1, t_2 and Coordinate must have definite value $q_K(t_1)$ and $q_K(t_2)$ respectively i.e.

$$\delta q_K(t_1) = \delta q_K(t_2) = 0$$

So $\sum_K \left[\frac{\partial L}{\partial \dot{q}_K} \delta q_K \right]_{t_1}^{t_2} = 0$

$$\int_{t_1}^{t_2} \sum_K \frac{\partial L}{\partial q_K} \delta q_K \, dt - \int_{t_1}^{t_2} \sum_K \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_K}\right) \delta q_K \, dt = 0$$

$$\sum_K \int_{t_1}^{t_2} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_K}\right) - \frac{\partial L}{\partial q_K} \right] \delta q_K \, dt = 0$$

We know that the Generalized Co-ordinates δq_K are independent of Each other then Coefficient of δq_K must Vanish

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_K}\right) - \frac{\partial L}{\partial q_K} = 0}$$

Where $K = 1, 2 \cdots n$ are the Generalized Coordinate

This is Lagrange's Equation of Motion

m-Imp.

* Principle of Least Action :-

This principal is based upon the Hamilton Variation principle in which the Variation in position and time Component is taken into account. This principle Can be Explain with the help of Hamilton Variation Principle in which the change in position Coordinate at the End points Remains Zero while the time Components does Not Equal to Zero which Can be Expressed as.

$$\boxed{A = \Delta \int_{t_1}^{t_2} \sum_{K} P_K \dot{q}_K \, dt = 0}$$

The motion of the particle is Explain for very short duration so it is known as Least Action principle which Can be Explain by a physical quantity known as Action.

Proof :- Let us Consider Hamilton's Principles function (or Action Integral) S is given by

$$S = \int_{t_1}^{t_2} L\,dt \quad \text{--- (1)}$$

Now Applying the Δ-Variation of S is

$$\Delta S = \Delta \int_{t_1}^{t_2} L\,dt$$

$$= \left[\delta + \Delta t \frac{d}{dt}\right] \int_{t_1}^{t_2} L\,dt \qquad \left[\Delta = \delta + \Delta t \frac{d}{dt}\right]$$

$$= \delta \int_{t_1}^{t_2} L\,dt + \int_{t_1}^{t_2} \Delta t\, d(L)$$

$$= \delta \int_{t_1}^{t_2} L\,dt + \left[L\,\Delta t \right]_{t_1}^{t_2}$$

$$= \int_{t_1}^{t_2} \sum_K \left[\underbrace{\frac{\partial L}{\partial q_K} \delta q_K + \frac{\partial L}{\partial \dot{q}_K} \delta \dot{q}_K}_{\text{middle Term ok2}} \right] dt + \left[L\,\Delta t \right]_{t_1}^{t_2} \quad \text{--- (2)}$$

In the present case $\delta q_K \neq 0$ at the end points

hence $\delta \int_{t_1}^{t_2} L\,dt$ is not Equal to zero

Now According to Lagrange's Equation we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_K} \right) - \frac{\partial L}{\partial q_K} = 0$$

$$\frac{\partial L}{\partial q_K} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_K} \right)$$

And $\delta \dot{q}_K = \frac{d}{dt}(\delta q_K)$

Now from Eq(2) the 1st Term of the Equation (middle) is solved in shown way i.e.

$$\frac{\partial L}{\partial q_K}\delta q_K + \frac{\partial L}{\partial \dot{q}_K} \delta \dot{q}_K = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_K} \right] \delta q_K + \frac{\partial L}{\partial \dot{q}_K} \frac{d}{dt}(\delta q_K)$$

$$\frac{\partial L}{\partial q_K}\delta q_K + \frac{\partial L}{\partial \dot{q}_K} \delta \dot{q}_K = \frac{d}{dt}\left[\underset{\mathrm{II}}{\frac{\partial L}{\partial \dot{q}_K}} \; \underset{\mathrm{I}}{\delta q_K} \right]$$

$$\frac{\partial L}{\partial q_K}\delta q_K + \frac{\partial L}{\partial \dot{q}_K} \delta \dot{q}_K = \frac{d}{dt}\left[p_K\, \delta q_K \right] \quad \text{--- (3)}$$

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we know that.

$$\Delta = \delta + \Delta t \frac{d}{dt}$$

$$\Delta q_K = \delta q_K + \Delta t \frac{\delta q_K}{dt}$$ [multiply q_K both side

$$\delta q_K = \Delta q_K - \Delta t \dot{q}_K$$

Now multiply both side by p_K we get

$$p_K \delta q_K = p_K \Delta q_K - p_K \dot{q}_K \Delta t$$

Now from Eq (3), putting the value of $p_K \delta q_K$ in eq (3) we get

$$\frac{\partial L}{\partial q_K} \delta q_K + \frac{\partial L}{\partial \dot{q}_K} \delta \dot{q}_K = \frac{d}{dt}\left[p_K \Delta q_K - p_K \dot{q}_K \Delta t\right]$$

$$\frac{\partial L}{\partial q_K} \delta q_K + \frac{\partial L}{\partial \dot{q}_K} \delta \dot{q}_K = \frac{d}{dt}\left[p_K \Delta q_K\right] - \frac{d}{dt}\left[p_K \dot{q}_K \Delta t\right] \quad \text{--- (4)}$$

Now putting the value of $\frac{\partial L}{\partial q_K} \delta q_K + \frac{\partial L}{\partial \dot{q}_K} \delta \dot{q}_K$

in Eq (2) we get

$$\Delta S = \Delta \int_{t_1}^{t_2} L \, dt = \int_{t_1}^{t_2} \sum_K \left[\frac{d}{dt}(p_K \Delta q_K) - \frac{d}{dt}(p_K \dot{q}_K \Delta t)\right] dt + \left[L \Delta t\right]_{t_1}^{t_2}$$

$$\Rightarrow \sum_K \int_{t_1}^{t_2} \left[d(p_K \Delta q_K) - d(p_K \dot{q}_K \Delta t)\right] + \left[L \Delta t\right]_{t_1}^{t_2}$$

$$\sum_K \left[p_K \Delta q_K\right]_{t_1}^{t_2} - \sum_K \left[p_K \dot{q}_K \Delta t\right]_{t_1}^{t_2} + \left[L \Delta t\right]_{t_1}^{t_2} \quad \text{--- (5)}$$

Now we know that $\Delta q_K = 0$ at the End points so $\left[p_K \Delta q_K\right]_{t_1}^{t_2} = 0$ putting this Value in Eq (5) we gets

$$\Delta S = \Delta \int_{t_1}^{t_2} L\, dt = 0 - \sum_K \left[p_K \dot{q}_K \Delta t\right]_{t_1}^{t_2} + \left[L \Delta t\right]_{t_1}^{t_2}$$

$$= \Delta \int_{t_1}^{t_2} L\, dt = \left[L \Delta t\right]_{t_1}^{t_2} - \sum_K \left[p_K \dot{q}_K \Delta t\right]_{t_1}^{t_2}$$

$$= \Delta \int_{t_1}^{t_2} L\, dt = \left[\left(L - \sum_K p_K \dot{q}_K\right) \Delta t\right]_{t_1}^{t_2} \quad \text{--- (6)}$$

We know that

$$H = \sum_K p_K \dot{q}_K - L$$

$$L + H = \sum_K p_K \dot{q}_K$$

$$L - \sum_K p_K \dot{q}_K = -H$$

putting this Value in Eq (6) we gets

$$\Delta \int_{t_1}^{t_2} L\, \Delta t = \left[-H \Delta t\right]_{t_1}^{t_2} \quad \text{--- (7)}$$

If we Restrict to System for which $\frac{dH}{dt} = 0$ and ~~two~~ to Variation for which H Remains Constants. we gets

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$$\Delta\int_{t_1}^{t_2} H\,dt = \int_{t_1}^{t_2} H\,d(\Delta t) = \left[H\,\Delta t\right]_{t_1}^{t_2}$$

putting the value of $\left[H\Delta t\right]_{t_1}^{t_2}$ in Eq (7) we get

$$\Delta\int_{t_1}^{t_2} L\,dt = -\Delta\int_{t_1}^{t_2} H\,dt$$

$$\Delta\int_{t_1}^{t_2} L\,dt + \Delta\int_{t_1}^{t_2} H\,dt = 0$$

$$\Delta\int_{t_1}^{t_2} \left[H+L\right] dt = 0 \quad \text{—— (8)}$$

But we know that

$$H = \sum_K p_K \dot{q}_K - L$$

$$H + L = \sum_K p_K \dot{q}_K$$

putting the value of $H+L$ in Eq (8) we get

$$\boxed{\Delta\int_{t_1}^{t_2} \sum_K p_K \dot{q}_K\,dt = 0}$$

This is known as principle of Least Action, the quantity $\int_{t_1}^{t_2} \sum_K p_K \dot{q}_K\,dt = W$

is called Hamilton characteristic function so the principle of least Action can be stated as

$$\boxed{\Delta W = \Delta\int_{t_1}^{t_2} \sum_K p_K \dot{q}_K\,dt = 0}$$

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