* Invariance of Poisson brackets with respect to Canonical Transformation

Sol^n

Poisson brackets are invariant under a Canonical transformation. It means mathematically

$$[X, Y]_{q,p} = [X, Y]_{Q,P} \quad \text{—— (1)}$$

we consider first the General Case.

$$[X, Y]_{Q,P} = \sum_i \left(\frac{\partial X}{\partial Q_i}\frac{\partial Y}{\partial P_i} - \frac{\partial X}{\partial P_i}\frac{\partial Y}{\partial Q_i} \right)$$

$$\Rightarrow \sum_{ij} \left\{ \frac{\partial X}{\partial Q_i}\left(\frac{\partial Y}{\partial q_j}\frac{\partial q_j}{\partial P_i} + \frac{\partial Y}{\partial p_j}\frac{\partial p_j}{\partial P_i} \right) - \frac{\partial X}{\partial P_i}\left(\frac{\partial Y}{\partial q_j}\frac{\partial q_j}{\partial Q_i} + \frac{\partial Y}{\partial p_j}\frac{\partial p_j}{\partial Q_i} \right) \right\}$$

$$= \sum_j \frac{\partial Y}{\partial q_j} \sum_i \left(\frac{\partial X}{\partial Q_i}\frac{\partial q_j}{\partial P_i} - \frac{\partial X}{\partial P_i}\frac{\partial q_j}{\partial Q_i} \right) + \sum_j \frac{\partial Y}{\partial p_j} \sum_i \left(\frac{\partial X}{\partial Q_i}\frac{\partial p_j}{\partial P_i} - \frac{\partial X}{\partial P_i}\frac{\partial p_j}{\partial Q_i} \right)$$

$$\Rightarrow \sum_j \left\{ \frac{\partial Y}{\partial q_j}[X, q_j]_{Q,P} + \frac{\partial Y}{\partial p_j}[X, p_j]_{Q,P} \right\} \quad \text{—— (2)}$$

Further

$$[X, q_j]_{Q,P} = -[q_j, X]_{Q,P}$$

$$= -\sum_m \left\{ \frac{\partial q_j}{\partial Q_m}\frac{\partial X}{\partial P_m} - \frac{\partial q_j}{\partial P_m}\frac{\partial X}{\partial Q_m} \right\}$$

$$= -\sum_{mk} \left\{ \frac{\partial q_j}{\partial Q_m}\left(\frac{\partial X}{\partial q_k}\frac{\partial q_k}{\partial P_m} + \frac{\partial X}{\partial p_k}\frac{\partial p_k}{\partial P_m} \right) - \frac{\partial q_j}{\partial P_m}\left(\frac{\partial X}{\partial q_k}\frac{\partial q_k}{\partial Q_m} + \frac{\partial X}{\partial p_k}\frac{\partial p_k}{\partial Q_m} \right) \right\}$$

* Invariance of Poisson bracket with respect to Canonical Transformation

Sol^n:-

Poisson bracket are invariant under a canonical transformation. It means mathematically

$$[X, Y]_{q,p} = [X, Y]_{Q,P} \quad \text{—} (1)$$

we consider first the general case.

$$[X, Y]_{Q,P} = \sum_i \left(\frac{\partial X}{\partial Q_i}\frac{\partial Y}{\partial P_i} - \frac{\partial X}{\partial P_i}\frac{\partial Y}{\partial Q_i} \right)$$

$$\Rightarrow \sum_{i,j} \left\{ \frac{\partial X}{\partial Q_i}\left(\frac{\partial Y}{\partial q_j}\frac{\partial q_j}{\partial P_i} + \frac{\partial Y}{\partial p_j}\frac{\partial p_j}{\partial P_i} \right) - \frac{\partial X}{\partial P_i}\left(\frac{\partial Y}{\partial q_j}\frac{\partial q_j}{\partial Q_i} + \frac{\partial Y}{\partial p_j}\frac{\partial p_j}{\partial Q_i} \right) \right\}$$

$$= \sum_j \frac{\partial Y}{\partial q_j} \sum_i \left(\frac{\partial X}{\partial Q_i}\frac{\partial q_j}{\partial P_i} - \frac{\partial X}{\partial P_i}\frac{\partial q_j}{\partial Q_i} \right) + \sum_j \frac{\partial Y}{\partial p_j} \sum_i \left(\frac{\partial X}{\partial Q_i}\frac{\partial p_j}{\partial P_i} - \frac{\partial X}{\partial P_i}\frac{\partial p_j}{\partial Q_i} \right)$$

$$\Rightarrow \sum_j \left\{ \frac{\partial Y}{\partial q_j}[X, q_j]_{Q,P} + \frac{\partial Y}{\partial p_j}[X, p_j]_{Q,P} \right\} \quad \text{—} (2)$$

further

$$[X, q_j]_{Q,P} = -[q_j, X]_{Q,P}$$

$$= -\sum_m \left\{ \frac{\partial q_j}{\partial Q_m}\frac{\partial X}{\partial P_m} - \frac{\partial q_j}{\partial P_m}\frac{\partial X}{\partial Q_m} \right\}$$

$$= -\sum_{m,k} \left\{ \frac{\partial q_j}{\partial Q_m}\left(\frac{\partial X}{\partial q_k}\frac{\partial q_k}{\partial P_m} + \frac{\partial X}{\partial p_k}\frac{\partial p_k}{\partial P_m} \right) - \frac{\partial q_j}{\partial P_m}\left(\frac{\partial X}{\partial q_k}\frac{\partial q_k}{\partial Q_m} + \frac{\partial X}{\partial p_k}\frac{\partial p_k}{\partial Q_m} \right) \right\}$$

$$= -\sum_K \left\{ \frac{\partial X}{\partial q_K} \sum_m \left(\frac{\partial q_j}{\partial Q_m}\frac{\partial q_K}{\partial P_m} - \frac{\partial q_j}{\partial P_m}\frac{\partial q_K}{\partial Q_m} \right) + \frac{\partial X}{\partial p_K} \sum_m \left(\frac{\partial q_j}{\partial Q_m}\frac{\partial p_K}{\partial P_m} - \frac{\partial q_j}{\partial P_m}\frac{\partial p_K}{\partial Q_m} \right) \right\}$$

$$= - \sum_K \left\{ \frac{\partial X}{\partial q_K} [q_j, q_K]_{Q,P} + \frac{\partial X}{\partial p_K} [q_j, p_K]_{Q,P} \right\}$$

$$= - \sum_K \frac{\partial X}{\partial p_K} \delta_{jK} = - \frac{\partial X}{\partial p_j} \quad \text{--- (4)}$$

Similarly

$$[X, p_j]_{Q,P} = \frac{\partial X}{\partial q_j}$$

Putting Eq (3) & (4) in Eq (2) we get

$$[X, Y]_{Q,P} = \sum_j \left\{ -\frac{\partial Y}{\partial q_j}\frac{\partial X}{\partial p_j} + \frac{\partial Y}{\partial p_j}\frac{\partial X}{\partial q_j} \right\}$$

$$- \frac{\partial X}{\partial p_i}\frac{\partial p_j}{\partial Q_i})$$

$$= [X, Y]_{q,p}$$

Which proves that Poisson brackets are invariant with respect to canonical transformation.

We have shown that a canonical transformation can be generated from function $F_1(q_j, Q_j, t)$

~~$F_2(q_j, Q_j, t)$~~ $F_2(q_j, P_j, t)$

$F_3(p_j, Q_j, t)$

$F_4(p_j, P_j, t)$.

Now in the case of F_1 generating function we have derived the

Relation

$$p_j = \frac{\partial F_1}{\partial q_j}$$

$$P_J = -\frac{\partial F_1}{\partial Q_J}$$

From these Relation:

$$\frac{\partial p_i}{\partial Q_l} = \frac{\partial^2 F_1}{\partial Q_l \partial q_j} = -\frac{\partial P_l}{\partial q_j} \quad \text{—— (5)}$$

Similarly in the case of $F_2(q_j, P_j, t)$ Generating function. the Relation are

$$p_j = \frac{\partial F_2}{\partial q_j}$$

$$Q_J = \frac{\partial F_2}{\partial P_J}$$

From these Relation we find that

$$\frac{\partial p_j}{\partial P_i} = \frac{\partial^2 F_2}{\partial P_i \partial q_j} = \frac{\partial Q_i}{\partial q_j} \quad \text{—— (6)}$$

Similarly from F_3 and F_4 Generating function we can find that

$$\frac{\partial q_j}{\partial Q_i} = \frac{\partial P_i}{\partial p_j} \quad \text{—— (7)}$$

and $$\frac{\partial q_j}{\partial P_i} = -\frac{\partial Q_i}{\partial p_j} \quad \text{—— (8)}$$

Now we write fundamental poisson bracket as.

$$[Q_i, P_j]_{q,p} = \sum_k \left(\frac{\partial Q_i}{\partial q_k} \frac{\partial P_j}{\partial p_k} - \frac{\partial Q_i}{\partial p_k} \frac{\partial P_j}{\partial q_k} \right)$$

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Applying the Results of Eq (7) & (5) we find

$$[Q_i, P_i]_{q,p} = \sum_K \left(\frac{\partial Q_i}{\partial q_K} \cdot \frac{\partial q_K}{\partial Q_j} + \frac{\partial Q_i}{\partial p_K} \cdot \frac{\partial p_K}{\partial Q_j} \right)$$

$$= \frac{\partial Q_i}{\partial Q_j} = \delta_{ij}$$

$$= [Q_i, P_j]_{Q,P}$$

And Similarly

$$[Q_i, Q_j]_{q,p} = 0 = [Q_i, Q_j]_{Q,P}$$

$$[P_i, P_j]_{q,p} = 0 = [P_i, P_j]_{Q,P}$$