$$q = -\frac{\partial f}{\partial p} = (e^{q}-1)^2 \sec^2 p$$

$$\frac{\partial f}{\partial p} = -\sec^2 p\,(e^{q}-1)^2$$

$$\partial f = -(e^{q}-1)^2 \int \sec^2 p \, dp$$

$$f = -(e^{q}-1)^2 \tan p$$

* Conservation of Energy: Jacobi's Integral :-

Let us consider

(i) a conservative system so that P.E is a function of coordinate only and not that of velocity

(ii) Constraints do not change with time i.e they are independents of time and consequently Equation of Transformation to generalised Coordinate do not involve time Explicitly and

(iii) L can be written as

$$L(q_j, \dot{q}_j)$$

the total time derivative of $L(q_j, \dot{q}_j)$ will be

$$\frac{\partial L}{\partial t} = \sum_j \frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial t} + \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial t}$$

putting $\frac{\partial L}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)$

$$\frac{dL}{dt} = \sum_j \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt}\right]$$

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$$= \sum_j \frac{d}{dt}\left(\underset{I}{\dot{q}_j} \underset{II}{\frac{\partial L}{\partial \dot{q}_j}}\right)$$

$$\frac{dL}{dt} = \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial T}{\partial \dot{q}_j}\right)$$

Since $L = T - V$
$V = 0$
$(\partial L = \partial T.)$

because for a conservative system

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} \text{ as } \frac{\partial V}{\partial \dot{q}_j} = 0$$

$$\text{also } p_j = \frac{\partial T}{\partial \dot{q}_j}$$

thus

$$\frac{dL}{dt} - \sum_j \frac{d}{dt}(\dot{q}_j p_j) = 0.$$

$$\frac{d}{dt}\left(\sum_j \dot{q}_j p_j - L\right) = 0$$

$$\sum_j \dot{q}_j p_j - L = \text{Constant}$$

$$\boxed{\sum_j \dot{q}_j p_j - L = J}$$

the Constant of motion J, is one of the first Integrals of Equation of motion and is by definition called Jacobi's Integral of the System

If all $\dot{q}_j$'s are substituted properly by function of p_j's then function J is nothing but Hamiltonian H.

$$H = \sum_j \dot{q}_j p_j - L = \text{Constant} \quad \text{(3)}$$

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K.E can be expressed as homogeneous quadratic function of generalised velocities.

$$T = \sum_{JK} a_{JK} \dot{q}_J \dot{q}_K$$

by Euler theorem

$$\sum_J q_J \frac{\partial f}{\partial q_J} = nf$$

here $n=2$

$$\sum_J \dot{q}_J \frac{\partial T}{\partial \dot{q}_J} = 2T$$

$$\sum \dot{q}_J P_J = 2T$$

from the Equation (3) we have

$$H = 2T - L = 2T - (T-V)$$

$$\boxed{H = T+V}$$

which shows that H Equals to the total Energy and is conserved.

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5 marks

Q A particle of mass m moves on a plane in the field of force given by

$$F = -\hat{r}\, K r \cos\theta$$

where K is constant and $\hat{r}$ is the radial unit vector.

(a) Will the angular momentum of the particle about the origin be conserved? Justify your statement.

(b) Obtain the differential equation of the orbit of the particle.

Soln:- Suppose (r, θ) are the polar coordinate of the particle of mass m in the given plane the K.E of the particle will be

$$T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) \quad \text{--- (1)}$$

(a) To show it. Let us write the Lagrange equation of motion in terms of θ. i.e

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_\theta \quad \text{--- (2)}$$

From Eq (1)

$$\left.\begin{aligned} \frac{\partial T}{\partial \dot{\theta}} &= m r^2 \dot{\theta} \\ \frac{\partial T}{\partial \theta} &= 0 \end{aligned}\right\}$$

Since the force is only radial $Q_\theta = 0$

So Eq (2) becomes

$$\frac{d}{dt}(m r^2 \dot{\theta}) = 0$$

$$\boxed{m r^2 \dot{\theta} = a \text{ (constant)}}$$

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The Left hand side of above Equation is the Angular momentum of the particle about the origin. And it is Conserved.

(b) differential Eq of the orbit is a Relation in Co-ordinate r And θ. we write the Lagrange Equation in Terms of Generalised Co-ordinate r i.e.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = Q_r \quad \text{---} (3)$$

from Eq (i)

$$\left.\begin{aligned} \frac{\partial T}{\partial \dot{r}} &= m\dot{r} \\ \frac{\partial T}{\partial r} &= mr\dot{\theta}^2 \end{aligned}\right\}$$

Also given $Q_r = F_r = -kr\cos\theta$

So the Eq (3) becomes

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 = -kr\cos\theta$$

$$\boxed{m\ddot{r} - mr\dot{\theta}^2 + kr\cos\theta = 0} \quad \text{---} (4)$$

this is the required differential Eq of the orbit of the particle

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$p = \text{constant}$ —— (3)

It means that momentum is always conserved in the hamiltonian of lagrangian when q is the cyclic coordinate. Hence we can say that if a coordinate is cyclic in the Lagrangian then it will be cyclic hamiltonian is also smooths.

* Lagrangian for a charged particle in an electromagnetic field: gyroscopic forces: Non conservative forces:-

Solution:

In an electromagnetic field the force on a particle of charge q is given by

$$f = q\left[E + \frac{1}{c}(V \times B)\right]$$

in which magnetic force $f = \frac{q}{c}(V \times B)$ is gyroscopic in nature.

Now we must express F in terms of magnetic vector and magnetic scalar potential A & ϕ respectively. thus.

$$B = \nabla \times A$$

$$E = -\nabla\phi - \frac{1}{c}\frac{dA}{dt}$$

Equation for force becomes.

$$F = q\left[-\nabla\phi - \frac{1}{c}\frac{dA}{dt} + \frac{1}{c}(V \times \nabla \times A)\right] \quad (1)$$

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Let us write the x- component of all Terms of Right hand side

$$(\nabla\phi)_x = \frac{\partial \phi}{\partial x}$$

And $(V\times\nabla\times A)_x = V_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - V_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)$

Now Adding And subtracting the Term $V_x \frac{\partial A_x}{\partial x}$ on the above Eq we get

So $V_x\frac{\partial A_x}{\partial x} + V_y\frac{\partial A_y}{\partial x} + V_z\frac{\partial A_z}{\partial x} - V_x\frac{\partial A_x}{\partial x} - V_y\frac{\partial A_x}{\partial y}$
$- V_z\frac{\partial A_x}{\partial z}$

$$\Rightarrow (V\times\nabla\times A)_x = \frac{\partial}{\partial x}(V\cdot A) - \left[V_x\frac{\partial A_x}{\partial x} + V_y\frac{\partial A_x}{\partial y} + V_z\frac{\partial A_x}{\partial z}\right] \quad (2)$$

For the total time derivative of A_x is

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + \left(V_x\frac{\partial A_x}{\partial x} + V_y\frac{\partial A_x}{\partial y} + V_z\frac{\partial A_x}{\partial z}\right)$$

$$\frac{dA_x}{dt} - \frac{\partial A_x}{\partial t} = V_x\frac{\partial A_x}{\partial x} + V_y\frac{\partial A_x}{\partial y} + V_z\frac{\partial A_x}{\partial z}$$

putting above in Eq (2) we get

$$(V\times\nabla\times A)_x = \frac{\partial}{\partial x}(V\cdot A) - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t}$$

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Eq (1) is now becomes.

$$F_x = q\left[-\frac{\partial \phi}{\partial x} - \frac{1}{c}\frac{\partial A_x}{\partial t} + \frac{1}{c}\left\{\frac{\partial}{\partial x}(v\cdot A) - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t}\right\}\right]$$

$$\Rightarrow \quad = q\left[-\frac{\partial}{\partial x}\left(\phi - \frac{1}{c}v\cdot A\right) - \frac{1}{c}\frac{dA_x}{dt}\right].$$

$\Rightarrow$ also $\frac{\partial}{\partial v_x}(v\cdot A) = A_x$; since $\frac{dA_x}{\partial v_x} = 0$

$$\text{so } F_x = q\left[-\frac{\partial}{\partial x}\left(\phi - \frac{1}{c}VA\right) + \frac{d}{dt}\left\{\frac{\partial}{\partial v_x}\left(-\frac{v\cdot A}{c}\right)\right\}\right]$$

Since ϕ is independent of velocity we can write the Term

$$\frac{\partial}{\partial v_x}\left(-\frac{v\cdot A}{c}\right) \text{ as } \frac{\partial}{\partial v_x}\left(\phi - \frac{1}{c}v\cdot A\right)$$

therefore

$$F_x = q\left[-\frac{\partial}{\partial x}\left(\phi - \frac{1}{c}v\cdot A\right) + \frac{d}{dt}\left\{\frac{\partial}{\partial x}\left(\phi - \frac{1}{c}v\cdot A\right)\right\}\right]$$

Let us put

$$U = q\phi - \frac{q}{c}v\cdot A$$

$$\boxed{F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}}\right)} \quad \text{--- (3)}$$

$$\text{Now } \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \quad \text{--- (4)}$$

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Writing Eq (3) in Generalised Co-ordinates we have

$$F_x = Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

And putting it in Eq (4) we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

$$\frac{d}{dt}\left(\frac{\partial (T-U)}{\partial \dot{q}_j}\right) - \frac{\partial (T-U)}{\partial q_j} = 0$$

which can be written as

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0}$$

where Lagrangian

$$L = T - U$$

$$= T - q\phi + \frac{q}{c} v \cdot A.$$

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