Lagrange Equation for small oscillation-

Those molecule whose motion lies b/w the motion of stable And unstable molecules known as small molicule And whose oscillation is known as small oscillation which also lies b/w stable And unstable oscillation

A system of particle is said to be in Equalibrium if all particles remain in rest that is Net force acting on the particle is very high

Let us consider a Generalized force acting on each particle is given as

$$Q_j = -\left(\frac{\partial V}{\partial q_j}\right)_0 \quad \text{——} (1)$$

if we assume the Equalibrium to be stable then condition of Equalibrium can be written as

$$q_j = q_{0j} + u_j$$

where q_{0j} is the displacement of Generalized coordinate in stable position (It is fixed) u_j is the displacement of the Generalized co-ordinate from there stable possition (as Generalized co-ordinate)

Let potential Energy if the function of u_j, q_{0j} then in Expanded form It may be written as

$$V(q_1, q_2 -- q_n) = V(u_1, u_2 -- u_n)$$
$$= V(q_{01}, q_{02} - q_{0n})$$

In expanded form the above eq^n may be written as. (Taylor series's expansion)

$$V = V(q_{01}, q_{02} --- q_{0j}) + \sum_j \left(\frac{\partial V}{\partial q}\right)_0 u_j + \frac{1}{2}\sum_{jk}\left(\frac{\partial^2 V}{\partial q_j \partial q_k}\right) u_j u_k + ---$$

all the Condition in above Equation show unstable Condition Except the third Term then the potential energy Can be Express as

$$V = \frac{1}{2}\sum_{JK}\left(\frac{\partial^2 V}{\partial q_j \partial q_K}\right) u_j u_K$$

$$V = \frac{1}{2}\sum_{JK} V_{jK} u_j u_K \quad — (2)$$

$$V_{JK} = \frac{\partial^2 V}{\partial q_j \partial q_K}$$

Now from the definition of K.E

$$T = \frac{1}{2}\sum_{JK} m_{jK} \dot{u}_j \dot{u}_K \quad — (3)$$

where u_j, u_K and q_j and q_K are the displacements

$u_j = q_j$
$u_K = q_K$
$\dot{u}_j = \dot{q}_j$
$\dot{u}_K = \dot{q}_K$

than above eq also be written as

$$T = \frac{1}{2}\sum_{JK} m_{JK} \dot{q}_j \dot{q}_K \quad — (4)$$

$$T = \frac{1}{2}\sum_{JK} T_{JK} \dot{q}_j \dot{q}_K \quad — (5)$$

from the definition of Lagrangian

$$L = T - V$$

$$L = \frac{1}{2}\sum_{JK} T_{JK} \dot{q}_j \dot{q}_K - \frac{1}{2}\sum_{JK} V_{JK} q_j q_K \quad — (6)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \quad (7)$$

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{1}{2} T_{JK} \dot{q}_K$$

$$\frac{\partial L}{\partial q_j} = -\frac{1}{2} V_{JK} \cdot q_K$$

by Eq (7)

$$\frac{1}{2}\frac{d}{dt}(T_{JK}\dot{q}_K) + \frac{1}{2} q_K V_{JK} = 0$$

$$T_{JK}\ddot{q}_K + V_{JK} q_K = 0 \quad (8)$$

this is the required Expression for the Lagrange Eq which can be Expressed in matrix form for n co-ordinate as follows

$$\begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & & & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \\ \ddot{q}_n \end{bmatrix} + \begin{bmatrix} V_{11} & V_{12} & \cdots & V_{1n} \\ V_{21} & V_{22} & \cdots & V_{2n} \\ \vdots & & & \vdots \\ V_{n1} & V_{n2} & \cdots & V_{nn} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} = 0$$

In General form it may be written as

$$T\ddot{q} + Vq = 0$$

this is the required Expression of small oscillation in K.E + P.E.

* Normal Co-ordinate And Normal frequency of Vibration :-

The Generalized Co-ordinate which is used to describe the oscillation of single frequency is known as Normal frequency of Vibration And such as Generalized Coordinate is known as Normal Coordinate. the Transformation of motion of such Co-ordinate (Normal Coordinate) is described by a Term And in such Condition the displacement of the Normal Coordinate Can be Expressed as

$$L_J = \sum_{JK} q_{JK} \eta_K \text{ --- } (1)$$

Where q_{JK} is a Constant which change the Normal Co-ordinate into Another Coordinate. If I^{th} And J^{th} Molecules are same then the above Eq Can be written as

$$\mu = a\eta \text{ --- } (2)$$

Again the P.E of the molecule Can be Expressed as

$$V = \frac{1}{2} \sum_{JK} V_{JK} V_J V_K$$

$$V = \frac{1}{2} \sum_{JK} \mu_J V_{JK} \mu_K$$

Then by the Transformation matrix the above Equation reduced in the form of

$$V = \frac{1}{2} u^T V u$$

Also from Eq. (2) the above Equation reduced in the form of

$$V = \frac{1}{2} (an)^T V u$$

$$V = \frac{1}{2} a^T n^T V u$$

$$V = \frac{1}{2} n^T a^T V a n$$

And from the General formula we know that

$$a^T V a = \Omega$$

the above Equation reduced in the form of

$$V = \frac{1}{2} n^T \Omega n \quad (3)$$

Now K.E can be written as

$$T = \frac{1}{2} \sum_{JK} T_{JK} \dot{u}_J \dot{u}_K$$

$$T = \frac{1}{2} \sum_{JK} \dot{u}_J T_{JK} \dot{u}_K$$

Then Again by the matrix transformation the above Equation reduced in the form of

$$T = \frac{1}{2} \dot{u}^T T \dot{u}$$

Again from Eq. (2) the above Equation reduced in the form of

$$T = \frac{1}{2} \dot{n}^T a^T T a \dot{n}$$

Also from the General formula we know that

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$a^T T a = 1$

the above equation reduced in the form of

$$T = \frac{1}{2}\dot{\eta}^T \dot{\eta}$$

$$T = \frac{1}{2}\dot{\eta}_l^2 \quad \text{--- (4)}$$

where $\eta_l = \eta$

Now the lagrangian of the system can be expressed as

$$L = T - V$$

$$L = \frac{1}{2}\dot{\eta}_l^2 - \frac{1}{2}\eta^T \Omega \eta \quad \text{--- (5)}$$

The P.E of the small co-ordinates can be written in the quardiatic form and can be expressed as

$$V = \frac{1}{2}\eta^T \Omega \eta = \frac{1}{2}\omega_l^2 \eta_l^2 \quad \text{--- (6)}$$

then from Eq (5) & (6) we get

$$L = \frac{1}{2}(\dot{\eta}_l^2 - \omega_l^2 \eta_l^2) \quad \text{--- (7)}$$

Now from the lagrangian Equation of the motion we know that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_l}\right) - \frac{\partial L}{\partial \eta_l} = 0$$

Again from Eq (7) we get

$$\frac{\partial L}{\partial \dot{\eta}_l} = \dot{\eta}_l \text{ And } \frac{\partial L}{\partial \eta_l} = -\omega_l^2 \eta_l$$

then the above equation can be written as

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$$\frac{d}{dt}(\dot{\eta}_l) + \omega_l^2 \eta_l = 0$$

$$\boxed{\ddot{\eta}_l + \omega_l^2 \eta_l = 0}$$

where ω_l is the frequency of vibration And η is the Normal Coordinate for different Coordinate.

$$\ddot{\eta}_1 + \omega_1^2 \eta_1 = 0$$

$$\ddot{\eta}_2 + \omega_2^2 \eta_2 = 0$$

$$\vdots$$

$$\ddot{\eta}_l + \omega_l^2 \eta_l = 0$$

these are the Equation of Vibration of different molecules.

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